

An equation on random variables and systems of fermions

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Abstract

In this paper, we consider an equation on random variables which can be reduced to the equation which describes the evolution of systems of fermions. We give some results of well-posedness for this equation on the spheres and torus of dimension 2 and 3 and on the Euclidean space. We give results of scattering and blow-up on the Euclidean depending on if the equation is defocusing or focusing. We interpret the results in terms of the evolution of fermions.

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1 Motivations

In this paper, we present an equation on random variables related to systems of fermions. This section is dedicated to presenting this equation and explaining its relation to equations derived from many-body quantum physics. We consider that, under sufficient assumptions, a system of fermions should behave according to

$$i\partial_t \gamma = [-\Delta + w * \rho_\gamma, \gamma]$$

where γ is a non negative bounded integral operator with kernel $\gamma(y, x)$, where ρ_γ is the multiplication by $\gamma(x, x)$, and $[\cdot, \cdot]$ is the commutator. The map w may be a Dirac delta. This equation has been studied in [6, 7, 13, 21, 22, 30].

The interest is that the equation on random variable closely resembles the cubic Schrödinger equation, and the theory of Schrödinger equations only has to be adapted to random variables to provide results, which are eventually turned into properties for the systems of fermions.

In Sections 2, 3, 4, 5, we use previously existing techniques about the cubic Schrödinger equation and adapt them to random variables. In Section 6, we give and discuss corollaries of the previous sections for systems of fermions.

1.1 Dynamics of a system of fermions

Before describing the dynamics of a system of fermions, we start with the better known Bose-Einstein condensate.

A system of N bosons may be described by a wave function $\Psi(x_1, \dots, x_N)$. from \mathbb{R}^{3N} to \mathbb{C} . It satisfies under certain conditions the Schrödinger equation

$$i\partial_t \Psi = - \sum_{i=1}^N \Delta_{x_i} \Psi + \sum_{i \neq j} w_T(x_i - x_j) \Psi$$

where Δ_{x_i} is the laplacian with respect to the variable x_i and is related to the kinetic energy, and w_T is related to the interaction between particles and depends on the temperature T .

When one lowers the temperature and takes a large number of particles, the system becomes a Bose-Einstein condensate, and under a mean-field approximation, one writes $\Psi(x_1, \dots, x_N) = \prod_j u(x_j)$ with u satisfying an equation of the form :

$$i\partial_t u = -\Delta u + w * |u|^2 u.$$

This approximation is motivated by the fact that bosons are exchangeable particles, in the sense that Ψ is symmetric, that is

$$\Psi(x_{\sigma(1)}, \dots, x_{\sigma(N)}) = \Psi(x_1, \dots, x_N)$$

for all permutations σ . The derivation of Bose-Einstein dynamics from many-body quantum mechanics is a vast subject in the literature, see for instance [1, 15, 17, 19, 20, 23, 24, 26].

Let us now consider a system of fermions. It is described by a wave function Ψ satisfying the same kind of dynamics as a system of bosons. But since we are dealing with fermions, Ψ is anti-symmetric, that is

$$\Psi(x_{\sigma(1)}, \dots, x_{\sigma(N)}) = \varepsilon(\sigma)\Psi(x_1, \dots, x_N)$$

where $\varepsilon(\sigma)$ is the signature of the permutation σ . This is the Pauli principle. If one writes

$$\Psi(x_1, \dots, x_N) = \frac{1}{\sqrt{n!}} \sum_{\sigma} \varepsilon(\sigma) \prod_{j=1}^N u_{\sigma(j)}(x_j)$$

where u_j are orthonormal functions, then the dynamics of Ψ may be approached, under a mean-field approximation, by the Hartree-Fock equation :

$$\forall j = 1, \dots, N, \quad i\partial_t u_j = -\Delta u_j + w * \left(\sum_k |u_k|^2 \right) u_j.$$

Note that $\int \overline{u_k} u_j$ is a conserved quantity for this equation and hence the orthonormality is preserved under the flow. The derivation of the Hartree-Fock equation from many-body quantum mechanics may be found in [2, 3, 4, 14, 16].

Writing $\gamma = \sum_k |u_k \times u_k|$, where $|f \times g|$ is the operator such that

$$|f \times g|(v)(x) = \int \overline{g(y)} v(y) dy f(x),$$

we get that γ satisfies

$$i\partial_t \gamma = [-\Delta + w * \rho_\gamma, \gamma]$$

where $[\cdot, \cdot]$ is the commutator and ρ_γ is the diagonal of the integral kernel of γ , here $\rho_\gamma = \sum |u_k|^2$. We note that the number of particles N is equal to the trace of γ . One may consider this equation on self-adjoint integral operators γ such that $0 \leq \gamma \leq 1$. These are called density operators. One can then consider a more general setting for the systems of fermions. For instance, by not restricting γ to be a trace-class operator, one can consider infinite systems of particles. The stability of non-trace class stationary solutions is the subject of [21, 22], which inspired this paper.

1.2 Comparison with density operators

We present here the equation on random variables and explain how it is related to what has been said before.

We consider the equation on random variables :

$$i\partial_t X = -\Delta X + \mathbb{E}(|X|^2)X \tag{1}$$

on a probability space (Ω, \mathcal{A}, P) . We assume that X has values in $L_{\text{loc}}^2(M)$ where M is either \mathbb{S}^d , \mathbb{T}^d or \mathbb{R}^d .

We write

$$\langle f, g \rangle = \int_M \overline{f(x)} g(x) dx.$$

Proposition 1.1. *Let γ be the operator defined as*

$$\gamma = \int |X(\omega) \times X(\omega)| dP(\omega)$$

that is

$$\gamma(v) = \mathbb{E}(\langle X, v \rangle X).$$

Let ρ_γ be the diagonal of the integral kernel of γ . Then, γ solves the equation:

$$i\partial_t \gamma = [-\Delta + \rho_\gamma, \gamma]. \quad (2)$$

Remark 1.1. *This is the equation one can find in [21, 22] in the case $\omega = \delta$.*

Proof. Let v in the domain of definition of γ and let us differentiate $\gamma(v)$. We have

$$i\partial_t \gamma(v) = \mathbb{E}(\langle -i\partial_t X, v \rangle X) + \mathbb{E}(\langle X, v \rangle i\partial_t X)$$

and by replacing $i\partial_t X$ by its value, we get

$$i\partial_t \gamma(v) = \mathbb{E}(\langle \Delta X - \mathbb{E}(|X|^2)X, v \rangle X) + \mathbb{E}(\langle X, v \rangle (-\Delta X + \mathbb{E}(|X|^2)X)).$$

Because Δ and the multiplication by $\mathbb{E}(|X|^2)$ are self-adjoint, we get

$$\mathbb{E}(\langle \Delta X - \mathbb{E}(|X|^2)X, v \rangle X) = \mathbb{E}(\langle X, (\Delta - \mathbb{E}(|X|^2))v \rangle X) = \gamma((\Delta - \mathbb{E}(|X|^2))v).$$

As $\langle X, v \rangle$ depends only on the probability variable, we have

$$\langle X, v \rangle (-\Delta X + \mathbb{E}(|X|^2)X) = (-\Delta + \mathbb{E}(|X|^2))(\langle X, v \rangle X)$$

and since $-\Delta + \mathbb{E}(|X|^2)$ does not act on the random variable,

$$\mathbb{E}((- \Delta + \mathbb{E}(|X|^2))(\langle X, v \rangle X)) = (- \Delta + \mathbb{E}(|X|^2))\mathbb{E}(\langle X, v \rangle X) = (- \Delta + \mathbb{E}(|X|^2))(\gamma(v))$$

therefore

$$i\partial_t \gamma(v) = [-\Delta + \mathbb{E}(|X|^2), \gamma].$$

What is more, the integral kernel of γ is $\mathbb{E}(\overline{X}(y)X(x))$ and hence $\rho_\gamma(x) = \mathbb{E}(|X(x)|^2)$ which gives the result. \square

This proposition explains how one goes from a solution of (1) to a solution of (2). The following proposition explains how to pass from an initial datum for (2) to an initial datum for (1). Combining these two propositions and a global well-posedness property for (1), we get global existence for (2). Indeed, from an initial datum for (2), we get an initial datum for (1), which gives a global solution to (1), which is turned into a solution to (2)..

Proposition 1.2. *Let $s \geq 0$. Let γ_0 be a non negative trace class operator on $L^2(M)$ such that*

$$\text{Tr}((1 - \Delta)^s \gamma_0) < \infty.$$

There exists a probability space (Ω, \mathcal{A}, P) and a random variable on this space X_0 such that $X_0 \in L^2(\Omega, H^s(M))$ and for all $v \in L^2(M)$

$$\mathbb{E}(\langle X_0, v \rangle X_0) = \gamma_0(v).$$

Proof. As γ_0 is trace class and non-negative, there exists a sequence of non-negative numbers $(\alpha_n)_{n \in \mathbb{N}}$ and an orthonormal family of $L^2(M)$, $(e_n)_{n \in \mathbb{N}}$ such that

$$\gamma_0 = \sum_{n \in \mathbb{N}} \alpha_n |e_n \times e_n|$$

where $|e_n \times e_n|$ is the projection on $\mathbb{C}e_n$.

Let $(g_n)_{n \in \mathbb{N}}$ be a sequence of complex centred normalised independent Gaussian variables. Set

$$X_0 = \sum_{n \in \mathbb{N}} \sqrt{\alpha_n} g_n e_n.$$

Let $v \in L^2(M)$. We have

$$\mathbb{E}(\langle X_0, v \rangle X_0) = \sum_{k,l} \sqrt{\alpha_k \alpha_l} \langle e_k, v \rangle e_l \mathbb{E}(\bar{g}_k g_l)$$

and since $\mathbb{E}(\bar{g}_k g_l) = \delta_k^l$ where δ_k^l is the Kronecker symbol, we get

$$\mathbb{E}(\langle X_0, v \rangle X_0) = \sum_k \alpha_k \langle e_k, v \rangle e_k = \gamma_0(v).$$

Besides, we have by definition

$$\|X_0\|_{L^2(\Omega, H^s)}^2 = \mathbb{E}(\langle X_0, (1 - \Delta)^s X_0 \rangle)$$

and since $\text{Tr}(AB) = \text{Tr}(BA)$,

$$\|X_0\|_{L^2(\Omega, H^s)}^2 = \mathbb{E}(\text{Tr}(|X_0 \times X_0|(1 - \Delta)^s))$$

and by linearity of the trace and definition of X_0 ,

$$\|X_0\|_{L^2(\Omega, H^s)}^2 = \text{Tr}(\mathbb{E}(|X_0 \times X_0|)(1 - \Delta)^s) = \text{Tr}(\gamma_0(1 - \Delta)^s).$$

□

Remark 1.2. More generally, if γ_0 is a non-negative operator and X_0 is the Gaussian random field (see [25]) with covariance operator γ_0 then $\gamma_{X_0} = \gamma_0$.

1.3 Equilibria

The equation (2) has stationary states on \mathbb{R}^d , \mathbb{T}^d , and \mathbb{S}^d , or even on sufficiently symmetric spaces. By sufficiently symmetric spaces, we mean any manifold M such that there exists a transitive action of a group on M that leaves M invariant.

On \mathbb{R}^d and \mathbb{T}^d , all Fourier multipliers may be considered. Indeed, they commute with the Laplacian and their integral kernel is a function of $x - y$, making their diagonals constants, hence commuting with any operator.

On \mathbb{S}^d , one may consider functions of the Laplace-Beltrami operator. These operators commute with the Laplace-Beltrami operator and the diagonal of their kernels is also a constant. This is due to spherical symmetry and is explained later.

In this subsection, we present random variables related to these stationary states. What we obtain from this parallel are not stationary states but states whose laws are invariant under the flow of (1).

On the sphere \mathbb{S}^d For $n \in \mathbb{N}^*$, let $(e_{n,k})_{1 \leq k \leq N_n}$ be a L^2 basis of spherical harmonics of degree n , that is, $e_{n,k}$ satisfies

$$-\Delta_{\mathbb{S}^d} e_{n,k} = n(n+d-1)e_{n,k} = \lambda_n e_{n,k}$$

for all $k = 1, \dots, N_n$. The number N_n is the dimension of the spherical harmonics of degree n , it is equal to

$$N_n = \binom{n+d}{d} - \binom{n+d-2}{d} \sim \frac{2}{(d-1)!} n^{d-1}.$$

Let $(a_n)_{n \geq 1}$ be a sequence of complex numbers satisfying

$$\sum_{n \geq 1} n^{d+1} |a_n|^2 < \infty.$$

Let $(g_{n,k})_{n,k}$ be a sequence of independent complex Gaussian variables of law $\mathcal{N}(0, 1)$.

We set

$$Y_0 = \sum_{n,k} g_{n,k} a_n e_{n,k} \text{ and } m = \frac{1}{\text{vol}(\mathbb{S}^d)} \sum_{n \geq 0} N_n |a_n|^2$$

and finally

$$Y(t) = \sum_{n,k} g_{n,k} a_n e^{-it(\lambda_n + m)} e_{n,k}.$$

Proposition 1.3. *The random variable $Y(t)$ satisfies (1) and its law does not depend on t . Besides Y belongs to $L^2(\Omega, H^1(\mathbb{S}^d))$.*

Remark 1.3. *Even though Y is not a stationary solution, this makes Y a natural invariant or equilibrium for (1).*

To prove this proposition, we need the following lemma.

Lemma 1.4 ([27], Lemma 3.1 in [11]). *The quantity*

$$K_n(x) = \sum_{k=1}^{N_n} |e_{n,k}(x)|^2$$

does not depend on x and is equal to

$$\frac{N_n}{\text{vol}(\mathbb{S}^d)}.$$

As this lemma is crucial for the invariance of Y , we give some elements of its proof.

Proof. Let

$$\tilde{K}_n(x, y) = \sum_{k=1}^{N_n} e_{n,k}(x) \overline{e_{n,k}(y)}$$

be the integral kernel of the orthogonal projection on the spherical harmonics of degree n . Because the sphere is invariant under rotations, we have for every rotation R that $(e_{n,k} \circ R)_{1 \leq k \leq N_n}$ is also a L^2 orthonormal basis of the spherical harmonics of degree n . Hence, $\tilde{K}_n(Rx, Ry)$ is also the integral kernel of the orthogonal projection on the spherical harmonics of degree n . Thus, for all rotations R and all $x \in \mathbb{S}^d$

$$K_n(Rx) = \tilde{K}_n(Rx, Rx) = \tilde{K}_n(x, x) = K_n(x).$$

Let $x_0 \in \mathbb{S}^d$. For all $x \in \mathbb{S}^d$, there exists a rotation R such that $x = Rx_0$, hence

$$K_n(x) = K_n(Rx_0) = K_n(x_0)$$

and $K_n(x)$ does not depend on x .

Finally

$$K_n(x_0) = \frac{1}{\text{vol}(\mathbb{S}^d)} \int_{\mathbb{S}^d} K_n(x_0) dx = \frac{1}{\text{vol}(\mathbb{S}^d)} \int_{\mathbb{S}^d} K_n(x) dx.$$

And given the definition of K_n and the fact that $(e_{n,k})_{1 \leq k \leq N_n}$ is an orthonormal basis, we have

$$K_n(x_0) = \frac{1}{\text{vol}(\mathbb{S}^d)} \int_{\mathbb{S}^d} \sum_{k=1}^{N_n} |e_{n,k}(x)|^2 dx = \frac{N_n}{\text{vol}(\mathbb{S}^d)}$$

which concludes the proof. \square

Proof of Proposition 1.3. Let us compute $\mathbb{E}(|Y(x)|^2)$. Because of the independence of the Gaussian variables, we have

$$\mathbb{E}(|Y(x)|^2) = \sum_{n,k} |a_n|^2 |e_{n,k}(x)|^2.$$

We use the lemma to get

$$\mathbb{E}(|Y(x)|^2) = \sum_n |a_n|^2 K_n(x) = \sum_n |a_n|^2 \frac{N_n}{\text{vol}(\mathbb{S}^d)} = m.$$

We differentiate Y . We get

$$i\partial_t Y = \sum_{n,k} a_n g_{n,k} e^{-it(\lambda_n+m)} (\lambda_n + m) e_{n,k}$$

and since λ_n are the eigenvalues of $\Delta_{\mathbb{S}^d}$ and $m = \mathbb{E}(|Y(t, x)|^2)$,

$$i\partial_t Y = (-\Delta_{\mathbb{S}^d} + m)Y = (-\Delta_{\mathbb{S}^d} + \mathbb{E}(|Y(x)|^2))Y.$$

Therefore, Y solves (1).

The fact that the law of Y does not depend on t is due to the invariance of the law of a Gaussian under rotations.

Finally, we have

$$\|Y\|_{L^2(\Omega, H^1(\mathbb{S}^d))}^2 = \sum_{n,k} |a_n|^2 \lambda_n = \sum_n |a_n|^2 \lambda_n N_n$$

which converges since $|a_n|^2 \lambda_n N_n \sim n^{d+1} |a_n|^2$ up to a constant. \square

Remark 1.4. The random variable Y corresponds to $\gamma = f(-\Delta_{\mathbb{S}^d})$ with $|a_n|^2 = f(\lambda_n)$. Indeed, for all $v \in L^2(\mathbb{S}^d)$,

$$\mathbb{E}(\langle Y(t), v \rangle Y(t)) = \sum_{n,k} |a_n|^2 \langle e_{n,k}, v \rangle e_{n,k} = f(-\Delta)(v).$$

Note that the operator does not depend on t , which makes $f(-\Delta_{\mathbb{S}^d})$ a stationary state for (2).

On \mathbb{T}^d Let $(a_k)_{k \in \mathbb{Z}^d}$ be a sequence of complex numbers such that

$$\sum_{k \in \mathbb{Z}^d} (1 + |k|^2) |a_k|^2 < \infty$$

where $|k|^2 = \sum_i k_i^2$ and let (g_k) be a sequence of independent centred normalised and complex Gaussian variables. We set

$$Y_0(x) = \sum_{k \in \mathbb{Z}^d} a_k g_k e^{ikx}$$

with $kx = \sum_i k_i x_i$. Let $m = \sum_k |a_k|^2$, and

$$Y(t) = \sum_{k \in \mathbb{Z}^d} a_k e^{-it(k^2+m)} g_k e^{ikx}.$$

Proposition 1.5. *The random variable Y is a solution to (1) belonging to $L^2(\Omega, H^1(\mathbb{T}^d))$ whose law does not depend on t .*

Proof. We have that thanks to the independence of the g_k that

$$\mathbb{E}(|Y(t, x)|^2) = m$$

and

$$i\partial_t Y = (-\Delta + m)Y.$$

The law of Y does not depend on t as the law of Gaussian variable is invariant under rotations. The variable Y belongs to $L^2(\Omega, H^1(\mathbb{S}^d))$ since

$$\sum_{k \in \mathbb{Z}^d} (1 + |k|^2) |a_k|^2 < \infty$$

□

Remark 1.5. *The random variable Y corresponds to the Fourier multiplier γ by $|a_k|^2$, that is for all t , $\gamma_{Y(t)} = \gamma$, which makes γ a stationary state of (2).*

On \mathbb{R}^d Let W be a d -dimensional complex centred random Gaussian process such that for all $k = (k_1, \dots, k_d) \in \mathbb{R}^d$ and $k' = (k'_1, \dots, k'_d) \in \mathbb{R}^d$, we have

$$\mathbb{E}(W(k) \overline{W(k')}) = \begin{cases} 0 & \text{if there exists } j \in [1, d] \text{ such that } k_j k'_j < 0 \\ \prod_{j=1}^d \min(|k_j|, |k'_j|) & \text{otherwise.} \end{cases}$$

In other terms, $\mathbb{E}(dW(k) \overline{dW(k')}) = dk dk' \delta(k - k')$ and $W(0) = 0$ where δ is the Dirac delta in dimension d .

For more information about Gaussian processes, we refer to [25].

Let $f \in L^2(\mathbb{R}^d)$ such that $k \mapsto \sqrt{1 + |k|^2} f(k)$ belongs to $L^2(\mathbb{R}^d)$ and set Y_0 the random variable

$$Y_0(x) = \int f(k) e^{inx} dW(k).$$

where $k = (k_1, \dots, k_d)$ and $kx = \sum_i k_i x_i$. We write $m = \int_{\mathbb{R}^d} |f(k)|^2 dk$ and

$$Y(t, x) = \int e^{-i(k^2+m)t} f(k) e^{ikx} dW(k).$$

Proposition 1.6. *The random variable $Y(t, x)$ is a solution of (1) whose law does not depend on t .*

Proof. The random variable Y satisfies

$$i\partial_t Y = -\Delta Y + mY.$$

We have

$$\begin{aligned} \mathbb{E}(|Y(t, x)|^2) &= \mathbb{E}\left(\left|\int e^{-i(k^2+m)t} f(k) e^{ikx} dW(k)\right|^2\right) \\ &= \int |e^{-i(k^2+m)t} f(k) e^{ikx}|^2 dk = \int |f(k)|^2 dk = m. \end{aligned}$$

Hence Y satisfies (1). The law of Y does not depend on time because Gaussian variables are invariant under rotations. \square

Remark 1.6. *The random variable Y is a natural invariant in the sense of the law for the equation (1). Nevertheless, it is not in $L^2(\mathbb{R}^d)$, or in $H^1(\mathbb{R}^d)$ but in $L^2_{loc}(\mathbb{R}^d)$, and in $H^1_{loc}(\mathbb{R}^d)$. That means that one cannot use the usual arguments of continuity in the initial datum or scattering to prove the stability of Y . We comment this lack of localisation in Section 4.*

Remark 1.7. *The operator associated to Y is the Fourier multiplier by $|f(k)|^2$. Indeed,*

$$\mathbb{E}(\langle Y, v \rangle Y) = \mathbb{E}\left(\int dy v(y) \int \overline{dW(k) f(k)} e^{-iky} \int dW(k') f(k') e^{ik'x}\right)$$

which yields, since $\mathbb{E}(dW(k) \overline{dW(k')}) = dk \delta(k - k')$,

$$\mathbb{E}(\langle Y, v \rangle Y) = \int dk e^{ikx} |f(k)|^2 \hat{v}(k).$$

We sum up the parallels we have made in this section in the following table.

| | Operator level | Random variable level |
|---|---|---|
| Equation | $i\partial_t \gamma = [-\Delta + \rho_\gamma, \gamma]$ | $i\partial_t X = -\Delta X + \mathbb{E}(X ^2)X$ |
| Solution | $\gamma = \mathbb{E}(X \times X)$ | X |
| Initial datum | γ_0 | Gaussian field with covariance γ_0 |
| Compact initial datum | $\gamma_0 = \sum_n \alpha_n ^2 u_n \times u_n $ | $X_0 = \sum_n \alpha_n u_n g_n$ with $(g_n)_n$ i.i.d $\mathcal{N}(0, 1)$ |
| Possible condition on the initial datum | $\text{Tr}(\gamma_0(1 - \Delta)) < \infty$ | $X_0 \in L^2(\Omega, H^1)$ |
| Equilibrium on \mathbb{S}^d | $\sum_{n,k} a_n ^2 e_{n,k} \times e_{n,k} $ | $\sum_{n,k} a_n e^{-i(\lambda_n+m)t} e_{n,k} g_{n,k}$ |
| Equilibrium on \mathbb{T}^d | $\widehat{\gamma_0 \varphi}(k) = f(k) ^2 \hat{\varphi}(k)$ | $\sum_{k \in \mathbb{Z}^d} f(k) g_k \frac{e^{ikx}}{\sqrt{2\pi}} e^{-i(k^2+m)t}$ |
| Equilibrium on \mathbb{R}^d | $\widehat{\gamma_0 \varphi}(k) = f(k) ^2 \hat{\varphi}(k)$ | $\int_{\mathbb{R}^d} f(k) e^{ikx} e^{-i(k^2+m)t} dW_k$ |

Finally, we make one last remark, which is also the main subject of Section 4. In Section 4, we prove that (1) scatters when the initial datum is in $H^1(\mathbb{R}^3)$. This may explain why the equilibria are not localised. At least, it explains why they are not in $H^1(\mathbb{R}^3)$. Indeed, if $Y(t)$ is both an equilibrium and in $H^1(\mathbb{R}^3)$. Then, as it scatters it converges to the solution to the linear equation

$$i\partial_t X = -\Delta X$$

with an initial datum in $H^1(\mathbb{R}^3)$ and because of dispersion in \mathbb{R}^3 , $Y(t)$ goes to 0 in some sense as t goes to ∞ . But it is impossible, unless $Y(t)$ is almost surely 0, as the law of $Y(t)$ does not depend on t . We discuss this in more detail in Section 4.

1.4 Main results

Throughout the paper, we give some results derived from the Schrödinger equation's theory for the equation on X , that is (1), such as global well-posedness in the energy space in $\mathbb{T}^d, \mathbb{R}^d, \mathbb{S}^d$ for $d = 2, 3$ or scattering in $L^2(\Omega, H^1(\mathbb{R}^3))$ and in the case of the focusing equation, existence of blow-up solutions, but we choose to state here two results of global well-posedness for the equation on γ , that is (2).

Theorem 1. *Let $M \in \{\mathbb{S}^2, \mathbb{S}^3, \mathbb{T}^2, \mathbb{T}^3\}$. Let Σ be the set of non-negative operators γ on M such that $\text{Tr}((1 - \Delta)\gamma) < \infty$. Let d be the distance on Σ defined in Definition 6.6.*

The equation (2) is well-posed in $C(\mathbb{R}, \Sigma)$ in the sense that for all $\gamma_0 \in \Sigma$ there exists a solution of (2) with initial datum γ_0 in $C(\mathbb{R}, \Sigma)$, this solution is unique in $C(\mathbb{R}, \Sigma)$ and the flow thus defined is continuous in the initial datum.

Theorem 2. *Let $M \in \mathbb{R}^2, \mathbb{R}^3$. Let f be a bounded map on M such that $\langle k \rangle |f(k)| \in L^2$. Let γ_f be the Fourier multiplier by $|f|^2$. Let $\gamma_f^{1/2}$ be the Fourier multiplier by f . Let Σ_f be the set of non-negative operators γ on M such that there exists a square root of γ , $\gamma^{1/2}$ such that $Q = \gamma^{1/2} - \gamma_f^{1/2}$ satisfies $\text{Tr}(Q^*(1 - \Delta)Q) < \infty$. The set Σ_f with the distance d is a well-defined metric space.*

The equation (2) is well-posed in $C(\mathbb{R}, \Sigma_f)$ in the sense that for all $\gamma_0 \in \Sigma$ there exists a solution of (2) with initial datum γ_0 in $C(\mathbb{R}, \Sigma_f)$, this solution is unique in $C(\mathbb{R}, \Sigma)$ and the flow thus defined is continuous in the initial datum.

2 Well-posedness on \mathbb{S}^d and \mathbb{T}^d

The goal of this section is to prove global well-posedness results in the energy space.

2.1 Local well-posedness on \mathbb{S}^2 and \mathbb{T}^2

In this subsection, we explain why the equation (1) is locally well-posed in $L^2(\Omega, H^1(M_2))$ with $M_2 = \mathbb{S}^2$ or \mathbb{T}^2 . The proof is very similar to the deterministic case and we do not claim any novelty regarding these techniques. We include the proof to explain how to deal with the probability part. This analysis could be applied to more general manifolds of dimension 2, as the main tool, that is Strichartz estimates, holds in a more general setting than \mathbb{S}^2 or \mathbb{T}^2 . We refer to [8].

We recall that from [8], Strichartz estimates on the sphere implies a loss of derivative. We use the following Strichartz estimate : for all $f \in H^1(M_2)$ and with $S(t) = e^{it\Delta}$,

$$\|S(t)f\|_{L^3([-1,1], L^\infty(M_2))} \leq C\|S(t)f\|_{L^3([-1,1], W^{1/2,6}(M_2))} \leq C\|f\|_{H^1(M_2)}. \quad (3)$$

Let $T \leq 1$. We call \mathcal{L}_T the space $C([-T, T], H^1(M_2)) \cap L^3([-T, T], L^\infty(M_2))$ normed by

$$\|f\|_{\mathcal{L}_T} = \|f\|_{L^\infty([-T, T], H^1(M_2))} + \|f\|_{L^3([-T, T], L^\infty(M_2))}.$$

Proposition 2.1. *Let $R \geq 0$. There exists C such that with $T = \frac{1}{CR^6}$, and for all X_0 such that $\|X_0\|_{L^2(\Omega, H^1(M_2))} \leq R$ the equation (1) with initial datum X_0 has a unique solution X in $L^2(\Omega, \mathcal{L}_T)$, this solution is continuous in the initial datum and satisfies*

$$\|X\|_{L^2(\Omega, \mathcal{L}_T)} \leq CR.$$

Proof. We proceed with a contraction argument.

The Duhamel formulation of (1) is

$$X(t, x) = S(t)X_0 - i \int_0^t S(t - \tau) \left(\mathbb{E}(|X(\tau, x)|^2) \right) X(\tau, x) d\tau.$$

Let

$$A(X)(t, x) = S(t)X_0 - i \int_0^t S(t - \tau) \left(\mathbb{E}(|X(\tau, x)|^2) \right) X(\tau, x) d\tau.$$

We prove that A is contracting in a ball of radius CR .

Thanks to Strichartz estimate (3) and the invariance of the H^1 norm under $S(t)$, we get

$$\|A(X)\|_{\mathcal{L}_T} \leq C_1 \|X_0\|_{H^1(M_2)} + C_1 \int_{-T}^T \|\mathbb{E}(|X(\tau)|^2) X(\tau)\|_{H^1(M_2)} d\tau.$$

We estimate the H^1 norm by $\|\cdot\|_{L^2(M_2)} + \|\nabla \cdot\|_{L^2(M_2)}$. We have

$$\|\mathbb{E}(|X(\tau)|^2) X(\tau)\|_{L^2(M_2)} \leq \|\mathbb{E}(|X(\tau)|^2)\|_{L^\infty} \|X(\tau)\|_{L^2}.$$

As a consequence of Minkowski inequality, we have $\|\cdot\|_{L^p_{x^*}, L^q_{x^*}} \leq \|\cdot\|_{L^q_{x^*}, L^p_{x^*}}$ as long as $p \geq q$ hence

$$\|\mathbb{E}(|X(\tau)|^2)\|_{L^\infty} \leq \mathbb{E}(\|X(\tau)\|_{L^\infty}^2)$$

Integrating in time yields

$$\int_{-T}^T \|\mathbb{E}(|X(\tau)|^2) X(\tau)\|_{L^2(M_2)} d\tau \leq \mathbb{E} \left(\int_{-T}^T \|X(\tau)\|_{L^\infty}^2 d\tau \right) \|X\|_{L^\infty([-T, T], L^2(M_2))}$$

and using Hölder inequality in the integral in time gives

$$\int_{-T}^T \|\mathbb{E}(|X(\tau)|^2) X(\tau)\|_{L^2(M_2)} d\tau \leq T^{1/3} \mathbb{E}(\|X\|_{\mathcal{L}_T}^2) \|X\|_{\mathcal{L}_T}. \quad (4)$$

For the term including derivatives, we have

$$\nabla \left(\mathbb{E}(|X(\tau, x)|^2) X(\tau, x) \right) = 2 \operatorname{Re} \left(\mathbb{E}(\overline{X}(\tau, x) \nabla X(\tau, x)) \right) X(\tau, x) + \mathbb{E}(|X(\tau, x)|^2) \nabla X(\tau, x).$$

Hence, thanks to Hölder inequality on the mean value, we get

$$\left| \nabla \left(\mathbb{E}(|X|^2) X \right) \right| \leq 2 \|\nabla X(\tau, x)\|_{L^2(\Omega)} \|X\|_{L^2(\Omega)} |X| + \mathbb{E}(|X|^2) |\nabla X|.$$

We take the L^2 norm in space, we get

$$\|\nabla \left(\mathbb{E}(|X|^2) X \right)\|_{L^2(M_2)} \leq 2 \|\nabla X(\tau, x)\|_{L^2(M_2, L^2(\Omega))} \|X\|_{L^\infty(M_2, L^2(\Omega))} \|X\|_{L^\infty(M_2)} + \mathbb{E}(\|X\|_{L^\infty(M_2)}^2) \|X\|_{H^1}.$$

Integrating in time yields

$$\begin{aligned} \int_{-T}^T \|\nabla \left(\mathbb{E}(|X|^2) X \right)\|_{L^2(M_2)} d\tau \leq \\ T^{1/3} \left(2 \|\nabla X(\tau, x)\|_{L^\infty([-T, T], L^2(M_2, L^2(\Omega)))} \|X\|_{L^3([-T, T], L^\infty(M_2, L^2(\Omega)))} \|X\|_{L^3([-T, T], L^\infty(M_2))} \right. \\ \left. + \|X\|_{L^3([-T, T], L^2(\Omega, L^\infty(M_2)))}^2 \|X\|_{L^\infty([-T, T], H^1)} \right). \end{aligned}$$

We recall that $\|\cdot\|_{L^p_*, L^q_z} \leq \|\cdot\|_{L^q_*, L^p_z}$ as long as $p \geq q$ hence

$$\begin{aligned} \|\nabla X(\tau, x)\|_{L^\infty([-T, T], L^2(M_2, L^2(\Omega)))} &\leq \|\nabla X(\tau, x)\|_{L^2(\Omega, L^\infty([-T, T], L^2(M_2)))} \leq \|X\|_{L^2(\Omega, \mathcal{L}_T)} \\ \|X\|_{L^3([-T, T], L^\infty(M_2, L^2(\Omega)))} &\leq \|X\|_{L^2(\Omega, L^3([-T, T], L^\infty(M_2)))} \leq \|X\|_{L^2(\Omega, \mathcal{L}_T)} \\ \|X\|_{L^3([-T, T], L^2(\Omega, L^\infty(M_2)))} &\leq \|X\|_{L^2(\Omega, L^3([-T, T], L^\infty(M_2)))} \leq \|X\|_{L^2(\Omega, \mathcal{L}_T)}. \end{aligned}$$

We get

$$\int_{-T, T} \|\nabla (\mathbb{E}(|X(\tau)|^2))X(\tau)\|_{L^2(M_2)} \leq 3T^{1/3} \mathbb{E}(\|X\|_{\mathcal{L}_T}^2) \|X\|_{\mathcal{L}_T}. \quad (5)$$

Putting together (4) and (5) yields

$$\|A(X)\|_{\mathcal{L}_T} \leq C_1 \|X_0\|_{H^1(M_2)} + 4C_1 T^{1/3} \mathbb{E}(\|X\|_{\mathcal{L}_T}^2) \|X\|_{\mathcal{L}_T}.$$

Taking its L^2 norm in probability yields

$$\|A(X)\|_{L^2(\Omega, \mathcal{L}_T)} \leq C_1 R + 4C_1 T^{1/3} \|X\|_{L^2(\Omega, \mathcal{L}_T)}^3.$$

Hence, for $T \leq \frac{1}{(4C_1)^3 (2C_1 R)^6}$, the ball of $L^2(\Omega, \mathcal{L}_T)$ of radius $2C_1 R$ is stable under A .

We prove that A is contracting on this ball, we have

$$A(X_1) - A(X_2) = \int_0^t S(t - \tau) (\mathbb{E}(|X_1|^2)X_1 - \mathbb{E}(|X_2|^2)X_2) d\tau.$$

Since

$$\mathbb{E}(|X_1|^2)X_1 - \mathbb{E}(|X_2|^2)X_2 = \mathbb{E}(|X_1|^2)(X_1 - X_2) + \mathbb{E}(X_1 \overline{(X_1 - X_2)})X_2 + \mathbb{E}(\overline{X_2}(X_1 - X_2))X_2,$$

buying doing the same computations as previously, we get

$$\|A(X_1) - A(X_2)\|_{L^2(\Omega, \mathcal{L}_T)} \leq 4C_1 T^{1/3} (\|X_1\|_{L^2(\Omega, \mathcal{L}_T)}^2 + \|X_2\|_{L^2(\Omega, \mathcal{L}_T)}^2) \|X_1 - X_2\|_{L^2(\Omega, \mathcal{L}_T)}$$

thus on the ball of radius $2C_1 R$ we get

$$\|A(X_1) - A(X_2)\|_{L^2(\Omega, \mathcal{L}_T)} \leq 12C_1 (2C_1 R)^2 T^{1/3} \|X_1 - X_2\|_{L^2(\Omega, \mathcal{L}_T)}$$

and for $T < \frac{1}{(12C_1)^3 (2C_1 R)^6}$ the map A is contracting, which concludes the proof. \square

2.2 Local well-posedness on \mathbb{S}^3 and \mathbb{T}^3

In this subsection, we prove local well-posedness of (1) on \mathbb{T}^3 and \mathbb{S}^3 , relying on [5] and [10]. Once again, we adapt the techniques from these papers to deal with the probability space but we do not claim any novelty regarding the deterministic analysis. We remark that as opposed to dimension 2, one cannot use the same techniques for more general manifolds.

Let $M_3 = \mathbb{T}^3$ or \mathbb{S}^3 . Let $(e_k)_k$ be an orthonormal basis of $L^2(M_3)$ consisting in eigenvalues of the Laplace-Beltrami operator associated to the eigenvalue $(\lambda_k)_k$. For all $u \in L^2(M_3)$, let $u_k = \langle e_k, u \rangle$.

Let $X^{s,b}(M_3)$ be the Bourgain space induced by the norm

$$\|u\|_{X^{s,b}(M_3)}^2 = \sum_k \langle \lambda_k \rangle^s \|\langle \lambda_k + \tau \rangle \hat{u}_k(\tau)\|_{L^2(\mathbb{R}_\tau)}^2 \quad (6)$$

where \hat{u}_k is the Fourier transform in time of u_k .

Finally, for $T \leq 1$, let $X_T^{s,b}(M_3)$ be the Bourgain space induced by the norm

$$\|u\|_{X_T^{s,b}(M_3)} = \inf\{\|w\|_{X^{s,b}(M_3)} \mid w|_{[-T,T]} = u\} \quad (7)$$

Adapting the proof of Proposition 2.11 in [9] to M_3 as it is done in [10], one gets the following estimate

$$\left\| S(t)u_0 + \int_0^t S(t-\tau)F(x, \tau)d\tau \right\|_{X_T^{1,b}(M_3)} \lesssim \|u_0\|_{H^1(M_3)} + T^{1-b-b'}\|F\|_{X^{1,-b'}(M_3)} \quad (8)$$

for (b, b') satisfying $0 < b' < \frac{1}{2} < b$ and $b + b' < 1$.

From [5] for the torus and from [10] for the sphere, one can deduce the following trilinear estimate : there exists $(b, b') \in \mathbb{R}^2$ such that $0 < b' < \frac{1}{2} < b$ and $b + b' < 1$ such that

$$\|uvw\|_{X^{1,-b'}(M_3)} \lesssim \|u\|_{X^{1,b}(M_3)}\|v\|_{X^{1,b}(M_3)}\|w\|_{X^{1,b}(M_3)}. \quad (9)$$

We remark that the constant implied by (9) and the couple (b, b') may depend on M_3 .

Proposition 2.2. *Let $R \geq 0$. There exists C and $T = T(R)$, such that for all X_0 such that $\|X_0\|_{L^2(\Omega, H^1(M_3))} \leq R$ the equation (1) with initial datum X_0 has a unique solution X in $L^2(\Omega, X_T^{1,b}(M_3))$, this solution is continuous in the initial datum and satisfies*

$$\|X\|_{L^2(\Omega, X_T^{1,b}(M_3))} \leq CR.$$

Remark 2.1. *First, the $X^{s,b}$ norm controls the H^s norm. We also have persistence of higher regularity in the sense that if the initial datum belongs to $H^s(M_3)$ with $s > 1$, then the solution remains in $X_T^{s,b}(M_3)$ for a time T which depends only on the $H^1(M_3)$ norm of the initial datum. This is due to the fact that*

$$\left\| \int_0^t S(t-\tau)(|u(\tau)|^2 u(\tau))d\tau \right\|_{X_T^{s,b}(M_3)} \lesssim T^\alpha \|u\|_{X_T^{s,b}} \|u\|_{X_T^{1,b}}^2$$

for some positive α (see [5, 10]) which eventually leads to

$$\left\| \int_0^t S(t-\tau)(\mathbb{E}(|X(\tau)|^2)X(\tau))d\tau \right\|_{L^2(\Omega, X_T^{s,b}(M_3))} \lesssim T^\alpha \|X\|_{L^2(\Omega, X_T^{s,b}(M_3))} \|X\|_{L^2(\Omega, X_T^{1,b}(M_3))}^2.$$

Before we prove this proposition, we prove the following lemma.

Lemma 2.3. *For all $u, v, w \in L^2(\Omega, X^{1,b}(M_3))$, we have*

$$\|\mathbb{E}(uv)w\|_{L^2(\Omega, X^{1,-b'}(M_3))} \lesssim \|u\|_{L^2(\Omega, X^{1,b}(M_3))}\|v\|_{L^2(\Omega, X^{1,b}(M_3))}\|w\|_{L^2(\Omega, X^{1,b}(M_3))}.$$

Proof. We proceed by duality. Let h in the dual of $L^2(\Omega, X^{1,-b'}(M_3))$ that is $L^2(\Omega, X^{-1,b'}(M_3))$. We have

$$\langle \mathbb{E}(uv)w, h \rangle_{\Omega \times M_3} = \int_{\Omega} \int_{M_3} \mathbb{E}(u(x)v(x))w(\omega_1, x)h(\omega_1, x)dx dP(\omega_1)$$

where $\langle \cdot, \cdot \rangle_{\Omega \times M_3}$ is the inner product in $\Omega \times M_3$. We replace \mathbb{E} by an integral over Ω we get

$$\langle \mathbb{E}(uv)w, h \rangle_{\Omega \times M_3} = \int_{\Omega \times \Omega} \int_{M_3} u(\omega_2, x)v(\omega_2, x)w(\omega_1, x)h(\omega_1, x)dx dP(\omega_1)dP(\omega_2).$$

Using (9) for $u(\omega_2)$, $v(\omega_2)$ and $w(\omega_1)$, we get

$$|\langle \mathbb{E}(uv)w, h \rangle_{\Omega \times M_3}| \lesssim \int_{\Omega \times \Omega} \|u(\omega_2)\|_{X^{1,b}(M_3)} \|v(\omega_2)\|_{X^{1,b}(M_3)} \|w(\omega_1)\|_{X^{1,b}(M_3)} \|h(\omega_1)\|_{X^{-1,b'}}.$$

We can use Cauchy-Schwartz inequality on ω_1 and on ω_2 to get

$$|\langle \mathbb{E}(uv)w, h \rangle_{\Omega \times M_3}| \lesssim \|u\|_{L^2(\Omega, X^{1,b}(M_3))} \|v\|_{L^2(\Omega, X^{1,b}(M_3))} \|w\|_{L^2(\Omega, X^{1,b}(M_3))} \|h\|_{L^2(\Omega, X^{-1,b'})}.$$

which concludes the proof. \square

Proof of Proposition 2.2. Let

$$A(X) = S(t)X_0 + \int_0^t S(t-\tau) \mathbb{E}(|X|^2) X d\tau$$

such that the Duhamel formulation of (1) is

$$X = A(X).$$

We proceed with a contraction argument on $L^2(\Omega, X_T^{1,b}(M_3))$. We have, thanks to (8),

$$\|A(X)\|_{L^2(\Omega, X_T^{1,b}(M_3))} \lesssim \|X_0\|_{L^2(\Omega, H^1(M_3))} + T^{1-(b+b')} \|\mathbb{E}(|X|^2) X\|_{L^2(\Omega, X^{1,-b'}(M_3))}$$

and thanks to Lemma 2.3,

$$\|\mathbb{E}(|X|^2) X\|_{L^2(\Omega, X^{1,-b'}(M_3))} \lesssim \|X\|_{L^2(\Omega, X^{1,b}(M_3)_T)}^3.$$

For the same reasons

$$\|A(X_1) - A(X_2)\|_{L^2(\Omega, X_T^{1,b}(M_3))} \lesssim T^{1-b-b'} \|X_1 - X_2\|_{L^2(\Omega, X_T^{1,b}(M_3))} (\|X_1\|_{L^2(\Omega, X_T^{1,b}(M_3))} \|X_2\|_{L^2(\Omega, X_T^{1,b}(M_3))}).$$

Hence, as $b + b' < 1$, there exist C and $T(R)$ such that the ball of $L^2(\Omega, X_T^{1,b}(M_3))$ of radius CR is stable under A and such that A is contracting on this ball, which concludes the proof. \square

2.3 Global Well-posedness

Let $M = M_2$ or M_3 . In this subsection, we prove global well-posedness in $H^1(M)$ using energy methods.

Lemma 2.4. *Let*

$$\mathcal{E}(X) = \mathcal{E}_{kin}(X) + \mathcal{E}_{pot}(X) = \frac{1}{2} \int_{\Omega \times M} \bar{X}(1 - \Delta)X + \frac{1}{4} \int_M \mathbb{E}(|X|^2)^2.$$

The quantity \mathcal{E} is invariant under the flow of (1).

Proof. Thanks to an approximation argument and the persistence of higher regularity, see Remark 2.1, we can assume that X is regular enough so that the computations below are justified.

Let $X(t)$ be a solution of (1) and let us differentiate $\mathcal{E}(X(t))$. We have

$$\partial_t \mathcal{E}_{kin}(X(t)) = \operatorname{Re} \left(\int_{\Omega \times M} (\partial_t \bar{X}) X \right) + \operatorname{Re} \left(\int_{\Omega \times M} (\partial_t \bar{X}) (-\Delta X) \right).$$

Because X satisfies (1), we have

$$\operatorname{Re}\left(\int_{\Omega \times M} (\partial_t \bar{X}) X\right) = \operatorname{Im}\left(\int_{\Omega \times M} (-\Delta X + \mathbb{E}(|X|^2) \bar{X}) X\right)$$

and because of the imaginary part, this quantity is zero. Therefore,

$$\partial_t \mathcal{E}_{\text{kin}}(X(t)) = \operatorname{Im}\left(\int_{\Omega \times M} i \partial_t \bar{X} (-\Delta X)\right).$$

Differentiating $\mathcal{E}_{\text{pot}}(X(t))$ yields

$$\partial_t \mathcal{E}_{\text{pot}}(X(t)) = \frac{1}{2} \int_M \mathbb{E}(|X|^2) \partial_t (\mathbb{E}(|X|^2)).$$

As ∂_t and \mathbb{E} commute, we get

$$\partial_t \mathcal{E}_{\text{pot}}(X(t)) = \int_M \mathbb{E}(|X|^2) \mathbb{E}(\operatorname{Re}(\partial_t \bar{X} X))$$

and we write the second expectation as an integral in the sense that

$$\partial_t \mathcal{E}_{\text{pot}}(X(t)) = \int_{\Omega \times M} \mathbb{E}(|X|^2) \operatorname{Re}(\partial_t \bar{X} X)$$

which finally yields

$$\partial_t \mathcal{E}_{\text{pot}}(X(t)) = \operatorname{Re}\left(\int_{\Omega \times M} \partial_t \bar{X} \mathbb{E}(|X|^2) X\right) = \operatorname{Im}\left(\int_{\Omega \times M} i \partial_t \bar{X} \mathbb{E}(|X|^2) X\right).$$

Summing the derivatives of $\mathcal{E}_{\text{kin}}(X(t))$ and $\mathcal{E}_{\text{pot}}(X(t))$ gives

$$\partial_t \mathcal{E}(X(t)) = \operatorname{Im}\left(\int_{\Omega \times M} i \partial_t \bar{X} (-\Delta X + \mathbb{E}(|X|^2) X)\right)$$

and because of the imaginary part and the fact that X satisfies (1), we get $\partial_t \mathcal{E}(X(t)) = 0$, which concludes the proof. \square

Since \mathcal{E} controls the $L^2(\Omega, H^1(M))$ norm, we get the following proposition.

Proposition 2.5. *The equation (1) is globally well-posed in $L^2(\Omega, H^1(M))$.*

2.4 Continuity with regard to the initial datum, interpretation in terms of law

Remark 2.2. *The first thing one can remark is that we have continuity in the initial datum. Indeed, let X_1 and X_2 be the solutions of (1) with initial data $X_1(0)$ close to $X_2(0)$. Let R be the maximum of $\|X_1(0)\|_{L^2(\Omega, H^1(M))}$ and $\|X_2(0)\|_{L^2(\Omega, H^1(M))}$. Then, up to times of order R^{-6} for $M = M_2$ or R^{-N} for some N for M_3 , we have*

$$\|X_1(t) - X_2(t)\|_{L^2(\Omega, H^1(M))} \leq C \|X_1(0) - X_2(0)\|_{L^2(\Omega, H^1(M))}$$

with C independent from R .

Iterating this estimate for longer times, in view of the conservation of the energy \mathcal{E} yields estimates such as

$$\|X_1(t) - X_2(t)\|_{L^2(\Omega, H^1(M))} \leq C e^{cR^N(1+t)} \|X_1(0) - X_2(0)\|_{L^2(\Omega, H^1(M))}.$$

We note that the spaces we used in the local well-posedness were not optimal at least for M_2 , and one could probably reach finite time estimates for times of order $R^{-(4+\varepsilon)}$ for M_2 , $\varepsilon > 0$.

Remark 2.3. Let ρ_1^t for all $t \in \mathbb{R}$ be the law of, or the measure induced by, $X_1(t)$ and ρ_2^t be the law of $X_2(t)$. Let d_2 be the Wasserstein distance of order 2 on the measures on $H^1(M)$, that is

$$d_2(\mu_1, \mu_2) = \inf_{\mu \in \text{Marg}(\mu_1, \mu_2)} \left(\int \|u - v\|_{H^1}^2 d\mu(u, v) \right)^{1/2}$$

where μ_i for $i = 1, 2$ are measures on $H^1(M)$ such that $\int \|u\|_{H^1}^2 d\mu_i(u)$ are finite and $\text{Marg}(\mu_1, \mu_2)$ is the set of measures on $H^1(M) \times H^1(M)$ whose marginals are μ_1 and μ_2 . Let μ^t be the law of $(X_1(t), X_2(t))$. Since μ^t has for marginals the law of $X_1(t)$, that is ρ_1^t and the law of $X_2(t)$, that is ρ_2^t , we get that

$$d_2(\rho_1^t, \rho_2^t) \leq \left(\int \|u - v\|_{H^1}^2 d\mu^t(u, v) \right)^{1/2} = \|X_1(t) - X_2(t)\|_{L^2(\Omega, H^1(M))}$$

and therefore

$$d_2(\rho_1^t, \rho_2^t) \leq C e^{cR^N(1+|t|)} \|X_1(0) - X_2(0)\|_{L^2(\Omega, H^1(M))}.$$

Since this is true for all $X_1(0)$ and $X_2(0)$ with laws ρ_1^0 and ρ_2^0 , we get

$$d_2(\rho_1^t, \rho_2^t) \leq C e^{cR^N(1+|t|)} d_2(\rho_1^0, \rho_2^0)$$

which gives a continuity in the law of the initial datum.

If we replace the $X_2(t)$ by $Y(t)$, as the law of $Y(t)$, called ν , does not depend on t , we get that

$$d_2(\rho_1^t, \nu) \leq C e^{cR^N(1+|t|)} d_2(\rho_1^0, \nu)$$

which is a result of stability for ν under the flow of the equation (1).

Remark 2.4. We have what we could call orbital stability in the sense that as \mathcal{E} is conserved, $\mathcal{E}(X(t)) - \mathcal{E}(Y(t))$ does not depend on time. Nevertheless, $\mathcal{E}(X(t)) - \mathcal{E}(Y(t))$ does not control a norm of $X - Y$.

3 Well-posedness on the Euclidean space

In \mathbb{R}^d , the equilibria Y are not localised. In particular, the law of Y is invariant under translations. In this section, we prove the existence of dynamics for perturbations around Y which are localised, in the sense that we prove global well-posedness for solutions X of (1) that are written $X = Y + Z$ where Z is localised as $Z \in L^2(\Omega, H^1(\mathbb{R}^d))$.

3.1 Perturbed equation and local well-posedness for $d \leq 3$

We perturb Y . Let $X = Y + Z$ such that $Z(0) = Z_0$ is in $L^2(\Omega, H^1(\mathbb{R}^d))$. The random variable Z solves the equation

$$i\partial_t Z = (-\Delta + m)Z + \left(\mathbb{E}(|Z|^2) + 2\text{Re}(\mathbb{E}(\bar{Y}Z)) \right) (Y + Z). \quad (10)$$

Let $\mathcal{L}_T = L^p([-T, T], L^\infty(\mathbb{R}^d)) \cap C([-T, T], H^1(\mathbb{R}^d))$ with $p = 4\frac{d+1}{d(d-1)}$. We prove local well-posedness in $L^2(\Omega, \mathcal{L}_T)$. First, in dimension $d \leq 3$, we have Strichartz estimates in the sense that there exists C such that for all $g \in H^1$,

$$\|S(t)f\|_{\mathcal{L}_T} \leq \|f\|_{H^1}. \quad (11)$$

Indeed, for $d \leq 3$, $p > 2$ and with q such that

$$\frac{2}{p} + \frac{d}{q} = \frac{d}{2}$$

that is $q = d + 1$ or $\frac{1}{q} < \frac{1}{d}$, we get that thanks to Sobolev embeddings

$$\|S(t)g\|_{L^p, L^\infty} \leq C\|S(t)g\|_{L^p, W^{q,1}}$$

and thanks to Strichartz estimates and the commutation of the differential operator $D = \sqrt{1 - \Delta}$ and $S(t)$

$$\|S(t)g\|_{L^p, L^\infty} \leq C\|g\|_{H^1}.$$

Let

$$m_2 = \int k^2 |f(k)|^2 dk < \infty.$$

That means that for all t and all x , $\mathbb{E}(|\nabla Y(t, x)|^2) = m_2 < \infty$.

Proposition 3.1. *There exists C such that for all $Z_0 \in L^2(\Omega, H^1)$, with*

$$T = \min\left(1, \frac{1}{C(m + m_2)}, \frac{1}{C(\sqrt{m + m_2})\|Z_0\|_{L^2(\Omega, H^1)}^{p/(p-1)}}, \frac{1}{C\|Z_0\|_{L^2(\Omega, H^1)}^{(2p)/(p-2)}}\right)$$

the equation (10) with initial datum Z_0 admits a unique solution Z in $L^2(\Omega, \mathcal{L}_T)$. This solution satisfies

$$\|Z\|_{L^2(\Omega, \mathcal{L}_T)} \leq C\|Z_0\|_{L^2(\Omega, H^1)}.$$

Proof. By the Duhamel formula, Z is the fixed point of

$$A(Z) = S(t)Z_0 + \int_0^t S(t - \tau) \left((\mathbb{E}(|Z|^2) + 2\text{Re}(\mathbb{E}(\bar{Y}Z))) (Y + Z) \right) d\tau.$$

We proceed with a contraction argument.

Thanks to (11), we have

$$\|A(Z)\|_{\mathcal{L}_T} \leq C \left(\|Z_0\|_{H^1} + \int_0^T \|(\mathbb{E}(|Z|^2) + 2\text{Re}(\mathbb{E}(\bar{Y}Z))) (Y + Z)\|_{H^1} d\tau \right).$$

Taking the L^2 norm in probability yields

$$\|A(Z)\|_{L^2(\Omega, \mathcal{L}_T)} \leq C \left(\|Z_0\|_{L^2(\Omega, H^1)} + \int_0^T \|(\mathbb{E}(|Z|^2) + 2\text{Re}(\mathbb{E}(\bar{Y}Z))) (Y + Z)\|_{L^2(\Omega, H^1)} d\tau \right).$$

We use the definition of the $L^2(\Omega, H^1)$ norm of g such as the $L^2(\Omega \times \mathbb{R}^d)$ norm of g added to the $L^2(\Omega \times \mathbb{R}^d)$ norm of ∇g .

For the part not containing any derivative, we start by taking the L^2 norm in probability, which yields

$$\begin{aligned} \|(\mathbb{E}(|Z|^2) + 2\text{Re}(\mathbb{E}(\bar{Y}Z))) (Y + Z)\|_{L^2(\Omega \times \mathbb{R}^d)} &\leq (\mathbb{E}(|Z|^2) + 2m\sqrt{\mathbb{E}(|Z|^2)})(m + \sqrt{\mathbb{E}(|Z|^2)}) = \\ &= (\|Z\|_{L^2(\Omega)}^2 + 2m\|Z\|_{L^2(\Omega)})(m + \|Z\|_{L^2(\Omega)}). \end{aligned}$$

Taking the L^2 norm in space yields

$$\begin{aligned} \|(\mathbb{E}(|Z|^2) + 2\operatorname{Re}(\mathbb{E}(\bar{Y}Z)))(Y + Z)\|_{L^2(\Omega \times \mathbb{R}^d)} \leq \\ (\|Z\|_{L^\infty(\mathbb{R}^d, L^2(\Omega))} \|Z\|_{L^2(\mathbb{R}^d, L^2(\Omega))} + 2m \|Z\|_{L^2(\mathbb{R}^d, L^2(\Omega))})(m + \|Z\|_{L^\infty(\mathbb{R}^d, L^2(\Omega))}). \end{aligned}$$

As a consequence of Minkowski's inequality, we have $\|\cdot\|_{L^\infty(\mathbb{R}^d, L^2(\Omega \times \mathbb{R}^d))} \leq \|\cdot\|_{L^2(\Omega, L^\infty(\mathbb{R}^d))}$. Therefore,

$$\begin{aligned} \int_0^T \|(\mathbb{E}(|Z|^2) + 2\operatorname{Re}(\mathbb{E}(\bar{Y}Z)))(Y + Z)\|_{L^2(\Omega \times \mathbb{R}^d)} d\tau \leq \\ C \|Z\|_{L^2(\Omega, \mathcal{L}_T)} (T 2m^2 + T^{1-1/p} 3m \|Z\|_{L^2(\Omega, \mathcal{L}_T)} + T^{1-2/p} \|Z\|_{L^2(\Omega, \mathcal{L}_T)}^2). \end{aligned}$$

Let us deal with the part containing the derivatives. We look at the different terms under the integral. First, we differentiate

$$\nabla(\mathbb{E}(|Z|^2)Z) = E(|Z|^2) \nabla Z + 2\operatorname{Re}(\mathbb{E}(\bar{\nabla} Z Z))Z$$

and then we take the L^2 norm in probability, which yields

$$\|\nabla(\mathbb{E}(|Z|^2)Z)\|_{L^2(\Omega)} \leq 3\|Z\|_{L^2(\Omega)}^2 \|\nabla Z\|_{L^2(\Omega)}.$$

For the other terms, we get

$$\begin{aligned} \|\nabla(\mathbb{E}(|Z|^2)Y)\|_{L^2(\Omega)} &\leq 2\sqrt{m}\|Z\|_{L^2(\Omega)} \|\nabla Z\|_{L^2(\Omega)} + \|Z\|_{L^2(\Omega)}^2 \sqrt{m_2}, \\ \|\nabla(2\operatorname{Re}(\mathbb{E}(\bar{Y}Z))Z)\|_{L^2(\Omega)} &\leq 2\sqrt{m_2}\|Z\|_{L^2(\Omega)}^2 + 4\sqrt{m}\|Z\|_{L^2(\Omega)} \|\nabla Z\|_{L^2(\Omega)}, \\ \|\nabla(2\operatorname{Re}(\mathbb{E}(\bar{Y}Z))Y)\|_{L^2(\Omega)} &\leq 4\sqrt{mm_2}\|Z\|_{L^2(\Omega)} + 2m \|\nabla Z\|_{L^2(\Omega)}. \end{aligned}$$

Summing up all these terms separating the ones containing derivatives of Z and the other ones gives

$$\begin{aligned} \|\nabla((\mathbb{E}(|Z|^2) + 2\operatorname{Re}(\mathbb{E}(\bar{Y}Z)))(Y + Z))\|_{L^2(\Omega)} &\leq \|\nabla Z\|_{L^2(\Omega)} (3\|Z\|_{L^2(\Omega)}^2 + 6\sqrt{m}\|Z\|_{L^2(\Omega)} + 2m) \\ &\quad + \|Z\|_{L^2(\Omega)} (3\sqrt{m_2}\|Z\|_{L^2(\Omega)} + 4\sqrt{mm_2}). \end{aligned}$$

We remark that $\|Z\|_{L^2(\Omega \times \mathbb{R}^d)} \leq \|Z\|_{L^2(\Omega, H^1)}$. Hence, taking the L^2 norm in space in the previous inequality gives

$$\begin{aligned} \|\nabla((\mathbb{E}(|Z|^2) + 2\operatorname{Re}(\mathbb{E}(\bar{Y}Z)))(Y + Z))\|_{L^2(\Omega \times \mathbb{R}^d)} \leq \\ \|Z\|_{L^2(\Omega, H^1)} (3\|Z\|_{L^2(\Omega, L^\infty(\mathbb{R}^d))}^2 + (6\sqrt{m} + 3\sqrt{m_2})\|Z\|_{L^2(\Omega, L^\infty(\mathbb{R}^d))} + 2m + 4\sqrt{mm_2}). \end{aligned}$$

Integrating in time yields

$$\begin{aligned} \int_0^T \|\nabla((\mathbb{E}(|Z|^2) + 2\operatorname{Re}(\mathbb{E}(\bar{Y}Z)))(Y + Z))\|_{L^2(\Omega \times \mathbb{R}^d)} d\tau \leq \\ \|Z\|_{L^2, \mathcal{L}_T} (3T^{1-2/p} \|Z\|_{L^2(\Omega, \mathcal{L}_T)}^2 + (6\sqrt{m} + 3\sqrt{m_2})T^{1-1/p} \|Z\|_{L^2(\Omega, \mathcal{L}_T)} + (2m + 4\sqrt{mm_2})T). \end{aligned}$$

Going back to $A(Z)$, we have the estimate

$$\|A(Z)\|_{L^2(\Omega, \mathcal{L}_T)} \leq C' \left(\|Z_0\|_{L^2(\Omega, H^1)} + \|Z\|_{L^2(\Omega, \mathcal{L}_T)} \left(T(4m + 4\sqrt{mm_2}) + T^{1-1/p}(9\sqrt{m} + 3\sqrt{m_2})\|Z\|_{L^2(\Omega, \mathcal{L}_T)} + 4T^{1-2/p}\|Z\|_{L^2(\Omega, \mathcal{L}_T)}^2 \right) \right).$$

In conclusion if $\|Z\|_{L^2(\Omega, \mathcal{L}_T)} \leq 2C'\|Z_0\|_{L^2(\Omega, H^1)}$ then with

$$T = \min \left(1, \frac{1}{C(m + m_2)}, \frac{1}{C(\sqrt{m} + \sqrt{m_2})\|Z_0\|_{L^2(\Omega, H^1)}^{p/(p-1)}}, \frac{1}{C\|Z_0\|_{L^2(\Omega, H^1)}^{(2p)/(p-2)}} \right)$$

for a constant C big enough, we have

$$\|A(Z)\|_{L^2(\Omega, \mathcal{L}_T)} \leq 2C'\|Z_0\|_{L^2(\Omega, H^1)}$$

which means that the ball of $L^2(\Omega, \mathcal{L}_T)$ of radius $2C'\|Z_0\|_{L^2(\Omega, H^1)}$ is stable under A .

For the same reasons, we get that A is contracting for appropriate times, which concludes the proof. \square

3.2 Global well-posedness in the energy space for $d \leq 3$

Proposition 3.2. *The equation (10) is globally well-posed in H^1 .*

Proof. We proceed with a modified energy method. Let

$$\begin{aligned} A &= \frac{1}{2} \int_{\Omega \times \mathbb{R}^d} \bar{Z}(m - \Delta)Z \\ B &= \frac{1}{4} \int_{\mathbb{R}^d} \mathbb{E}(|Z|^2)^2 \\ D &= \int_{\mathbb{R}^d} \mathbb{E}(|Z|^2) \text{Re} \mathbb{E}(\bar{Z}Y). \end{aligned}$$

Differentiating these quantities in time yields

$$\begin{aligned} \partial_t A &= \text{Re} \int_{\Omega \times \mathbb{R}^d} \partial_t \bar{Z}(m - \Delta)Z \\ \partial_t B &= \text{Re} \int_{\Omega \times \mathbb{R}^d} \partial_t \bar{Z} \mathbb{E}(|Z|^2)Z \\ \partial_t D &= \text{Re} \int_{\Omega \times \mathbb{R}^d} \partial_t \left(Z 2 \text{Re} \mathbb{E}(\bar{Z}Y) + Y \mathbb{E}(|Z|^2) \right) + \text{Re} \int_{\mathbb{R}^d} \mathbb{E}(|Z|^2) \mathbb{E}(\bar{Z} \partial_t Y). \end{aligned}$$

We deduce from that

$$\partial_t(A + B + D) = \text{Re} \int_{\Omega \times \mathbb{R}^d} \partial_t \bar{Z} \left(i \partial_t Z - Y 2 \text{Re} \mathbb{E}(\bar{Z}Y) \right) + \text{Re} \int_{\mathbb{R}^d} \mathbb{E}(|Z|^2) \mathbb{E}(\bar{Z} \partial_t Y).$$

Because of the real part we get

$$\text{Re} \int_{\Omega \times \mathbb{R}^d} \partial_t \bar{Z} \left(i \partial_t Z - Y 2 \text{Re} \mathbb{E}(\bar{Z}Y) \right) = -\text{Re} \int_{\Omega \times \mathbb{R}^d} \partial_t \bar{Z} Y 2 \text{Re} \mathbb{E}(\bar{Z}Y)$$

and replacing $\partial_t \bar{Z}$ by its value

$$-\operatorname{Re} \int_{\Omega \times \mathbb{R}^d} \partial_t \bar{Z} Y 2 \operatorname{Re} \mathbb{E}(\bar{Z} Y) = -\operatorname{Re} \int_{\Omega \times \mathbb{R}^d} i \left((m - \Delta) \bar{Z} + (E(|Z|^2 + 2 \operatorname{Re}(\bar{Z} Y))(\bar{Z} + \bar{Y})) \right) Y 2 \operatorname{Re} \mathbb{E}(\bar{Z} Y)$$

and again because of the real part

$$-\operatorname{Re} \int_{\Omega \times \mathbb{R}^d} \partial_t \bar{Z} Y 2 \operatorname{Re} \mathbb{E}(\bar{Z} Y) = -\operatorname{Re} \int_{\Omega \times \mathbb{R}^d} i \left((m - \Delta) \bar{Z} + (E(|Z|^2 + 2 \operatorname{Re}(\bar{Z} Y)) \bar{Z}) \right) Y 2 \operatorname{Re} \mathbb{E}(\bar{Z} Y).$$

Returning to A, B, D , we get

$$\partial_t(A + B + D) = -\operatorname{Re} \int_{\Omega \times \mathbb{R}^d} i \left((m - \Delta) \bar{Z} + (E(|Z|^2 + 2 \operatorname{Re}(\bar{Z} Y)) \bar{Z}) \right) Y 2 \operatorname{Re} \mathbb{E}(\bar{Z} Y) + \operatorname{Re} \int_{\mathbb{R}^d} \mathbb{E}(|Z|^2) \mathbb{E}(\bar{Z} \partial_t Y).$$

We estimate the different terms of the sum, we have

$$\begin{aligned} \left| -\operatorname{Re} \int_{\Omega \times \mathbb{R}^d} i Y 2 \operatorname{Re} \mathbb{E}(\bar{Z} Y) (m - \Delta) \bar{Z} \right| &\leq C(m, m_2) A \\ \left| -\operatorname{Re} \int_{\Omega \times \mathbb{R}^d} i Y 2 \operatorname{Re} \mathbb{E}(\bar{Z} Y) \mathbb{E}(|Z|^2) \bar{Z} \right| &\leq C(m) B \\ \left| -\operatorname{Re} \int_{\Omega \times \mathbb{R}^d} i Y 2 \operatorname{Re} \mathbb{E}(\bar{Z} Y) 2 \operatorname{Re}(\bar{Z} Y) \bar{Z} \right| &\leq C(m) A^{1/2} B^{1/2} \\ \left| \operatorname{Re} \int_{\mathbb{R}^d} \mathbb{E}(|Z|^2) \mathbb{E}(\bar{Z} \partial_t Y) \right| &\leq C(m, m_2) A^{1/2} B^{1/2}. \end{aligned}$$

The problem with this method is that $A + B + D$ does not control the H^1 norm. For this, we set

$$E = \frac{1}{2} \int_{\Omega \times \mathbb{R}^d} |Z|^2.$$

We have

$$|D| \leq \sqrt{m} B^{1/2} E^{1/2} \leq \frac{1}{2} B + 2mE.$$

Hence setting $\mathcal{E} = A + B + D + 2mE$, we get $\mathcal{E} \geq A + \frac{1}{2}B$. We prove now that $|\partial_t \mathcal{E}| \leq C\mathcal{E}$. Because of the previous computations

$$|\partial_t(A + B + D)| \leq C(m, m_2) \mathcal{E}.$$

We compute the derivative of E . We have

$$\partial_t E = \operatorname{Im} \int_{\Omega \times \mathbb{R}^d} \bar{Z} i \partial_t Z = \operatorname{Im} \int_{\Omega \times \mathbb{R}^d} \bar{Z} \left((m - \Delta) Z + (E(|Z|^2 + 2 \operatorname{Re}(\bar{Z} Y))(Z + Y)) \right)$$

and because of the imaginary part

$$\partial_t E = \operatorname{Im} \int_{\Omega \times \mathbb{R}^d} \bar{Z} (E(|Z|^2 + 2 \operatorname{Re}(\bar{Z} Y)) Y).$$

We get

$$|\partial_t E| \leq \sqrt{m} E^{1/2} B^{1/2} + mE \leq C(m) (A^{1/2} B^{1/2} + A) \leq C(m) \mathcal{E}.$$

In conclusion, we get a bound for \mathcal{E} and thus for A , the $L^2(\Omega, H^1)$ norm of the solution, which implies global existence. \square

3.3 Local well-posedness in dimension 4 for small initial data

In this subsection, we prove local well posedness for small initial data in H^1 in dimension 4. We use a contraction argument in

$$\mathcal{L}_T = L^2(\Omega, C([-T, T], H^1(\mathbb{R}^4))) \cap L^2(\Omega, L^3([-T, T], W^{1,3}(\mathbb{R}^4))).$$

Thanks to Strichartz estimates, there exists C such that for all $T \geq 0$, and all $g \in L^2(\Omega, H^1(\mathbb{R}^4))$, we have

$$\|S(t)g\|_{\mathcal{L}_T} \leq C\|g\|_{L^2(\Omega, H^1(\mathbb{R}^4))}. \quad (12)$$

Proposition 3.3. *There exists $\varepsilon > 0$ such that for all Z_0 satisfying $\|Z_0\|_{L^2(\Omega, H^1(\mathbb{R}^4))} \leq \varepsilon$, the equation (10) admits a unique solution Z in \mathcal{L}_T for*

$$T = \min\left(\frac{1}{C(\sqrt{m} + \sqrt{m_2})^3}, \frac{1}{C(m + m_2)}\right)$$

with C big enough. Besides there exists C such that

$$\|Z\|_{\mathcal{L}_T} \leq 2C\varepsilon$$

and Z depends continuously in Z_0 .

Proof. Let

$$A(Z) = S(t)Z_0 - i \int_0^t S(t-\tau) \left((\mathbb{E}(|Z|^2) + 2\text{Re}\mathbb{E}(\bar{Y}Z))(Y + Z) \right) d\tau.$$

The solution Z is the fixed point of A . We have, thanks to (12),

$$\|A(Z)\|_{\mathcal{L}_T} \leq C \int_{-T}^T \left\| (\mathbb{E}(|Z|^2) + 2\text{Re}\mathbb{E}(\bar{Y}Z))(Y + Z) \right\|_{L^2(\Omega, H^1)} d\tau$$

which yields by a triangle inequality

$$\begin{aligned} \|A(Z)\|_{\mathcal{L}_T} \leq C & \left(\|Z\|_{L^2(\Omega, H^1(\mathbb{R}^4))} + \int_{-T}^T \left(\|\mathbb{E}(|Z|^2)Z\|_{L^2(\Omega, H^1)} + \|\mathbb{E}(|Z|^2)Y\|_{L^2(\Omega, H^1)} + \right. \right. \\ & \left. \left. \|2\text{Re}\mathbb{E}(\bar{Y}Z)Z\|_{L^2(\Omega, H^1)} + \|2\text{Re}\mathbb{E}(\bar{Y}Z)Y\|_{L^2(\Omega, H^1)} \right) d\tau \right). \end{aligned}$$

We have since $\|fg\|_{H^s} \lesssim \|(D^s f)g\|_{L^2} + \|fD^s g\|_{L^2}$,

$$\|\mathbb{E}(|Z|^2)Z\|_{L^2(\Omega, H^1)} \lesssim \|Z\|_{L^2(\Omega)}^2 \|DZ\|_{L^2(\Omega)} \|Z\|_{L^2(\mathbb{R}^4)}.$$

Thanks to Hölder inequality, as $\frac{1}{3} + \frac{1}{6} = \frac{1}{2}$,

$$\|\mathbb{E}(|Z|^2)Z\|_{L^2(\Omega, H^1)} \lesssim \|Z\|_{L^{12}(\mathbb{R}^4, L^2(\Omega))}^2 \|DZ\|_{L^3(\mathbb{R}^4, L^2(\Omega))}$$

and as 12 and 3 are bigger than 2, we can exchange the order of the norms,

$$\|\mathbb{E}(|Z|^2)Z\|_{L^2(\Omega, H^1)} \lesssim \|Z\|_{L^2(\Omega, L^{12}(\mathbb{R}^4))}^2 \|DZ\|_{L^2(\Omega, L^3(\mathbb{R}^4))}$$

and since $W^{1,3}(\mathbb{R}^4)$ is embedded in $L^{12}(\mathbb{R}^4)$,

$$\|\mathbb{E}(|Z|^2)Z\|_{L^2(\Omega, H^1)} \lesssim \|Z\|_{L^2(\Omega, W^{1,3}(\mathbb{R}^4))}^3.$$

Integrating in time yields

$$\int_{-T}^T \|\mathbb{E}(|Z|^2)Z\|_{L^2(\Omega, H^1)} \lesssim \|Z\|_{\mathcal{L}_T}^3. \quad (13)$$

Using that $\mathbb{E}(|Y|^2) = m$ and $\mathbb{E}(|DY|^2) = m + m_2$, we get for the quadratic terms

$$\|\mathbb{E}(|Z|^2)Y\|_{L^2(\Omega, H^1(\mathbb{R}^4))} + \|2\operatorname{Re}\mathbb{E}(\bar{Y}Z))Z\|_{L^2(\Omega, H^1)} \lesssim (\sqrt{m} + \sqrt{m_2})\|Z\|_{L^2(\Omega, L^6(\mathbb{R}^4))}\|Z\|_{L^2(\Omega, W^{1,3}(\mathbb{R}^4))}$$

and as $W^{1,3}(\mathbb{R}^4)$ is embedded in $L^6(\mathbb{R}^4)$ and integrating in time, we get

$$\int_{-T}^T \|\mathbb{E}(|Z|^2)Y\|_{L^2(\Omega, H^1(\mathbb{R}^4))} + \|2\operatorname{Re}\mathbb{E}(\bar{Y}Z))Z\|_{L^2(\Omega, H^1)} \lesssim (\sqrt{m} + \sqrt{m_2})T^{1/3}\|Z\|_{\mathcal{L}_T}^2. \quad (14)$$

For the linear term we have

$$\|2\operatorname{Re}\mathbb{E}(\bar{Y}Z))Y\|_{L^2(\Omega, H^1)} \lesssim (m + m_2)\|Z\|_{L^2(\Omega, H^1)}$$

which gives

$$\int_{-T}^T \|2\operatorname{Re}\mathbb{E}(\bar{Y}Z))Y\|_{L^2(\Omega, H^1)} \lesssim (m + m_2)T\|Z\|_{\mathcal{L}_T}. \quad (15)$$

Summing (13), (14), (15), we get

$$\|A(Z)\|_{\mathcal{L}_T} \leq C(\|Z\|_{L^2(\Omega, H^1(\mathbb{R}^4))} + \|Z\|_{\mathcal{L}_T}^3 + (\sqrt{m} + \sqrt{m_2})T^{1/3}\|Z\|_{\mathcal{L}_T}^2 + (m + m_2)T\|Z\|_{\mathcal{L}_T}).$$

Assuming that $\|Z\|_{L^2(\Omega, H^1(\mathbb{R}^4))} \leq \varepsilon$ with ε such that

$$2C\varepsilon \leq \frac{1}{2\sqrt{C}}$$

and assuming

$$T = \min\left(\frac{1}{8^3(\sqrt{m} + \sqrt{m_2})^3}, \frac{1}{8(m + m_2)}\right),$$

we get that the ball of \mathcal{L}_T of radius $2C\varepsilon$ is stable under the map A .

What is more, for the same reasons, we get

$$\begin{aligned} \|A(Z_1) - A(Z_2)\|_{\mathcal{L}_T} &\leq C(\|Z_1\|_{\mathcal{L}_T}^2 + \|Z_2\|_{\mathcal{L}_T}^2 + \\ &\quad (\sqrt{m} + \sqrt{m_2})T^{1/3}(\|Z_1\|_{\mathcal{L}_T} + \|Z_2\|_{\mathcal{L}_T}) + (m + m_2)T)\|Z_1 - Z_2\|_{\mathcal{L}_T}). \end{aligned}$$

Hence, for

$$T = \min\left(\frac{1}{C(\sqrt{m} + \sqrt{m_2})^3}, \frac{1}{C(m + m_2)}\right)$$

with C big enough and ε small enough, we get that A is contracting, which ensures existence and uniqueness of the fix point.

Finally, if Z^1 is the solution of (10) with initial datum Z_0^1 in the ball of radius ε , we have

$$Z^1 = S(t)(Z_0^1 - Z_0) + A(Z^1)$$

thus

$$Z^1 - Z = S(t)(Z_0^1 - Z_0) + A(Z^1) - A(Z)$$

and as A is contracting,

$$\|Z^1 - Z\|_{\mathcal{L}_T} \lesssim \|Z_0^1 - Z_0\|_{L^2(\Omega, H^1)}.$$

□

4 Scattering and non-existence of localised equilibrium

By copying the method of Lewin and Sabin in [22], it may be possible to prove scattering properties for the perturbed Hartree equation :

$$i\partial_t Z = (m - \Delta)Z + w * (\mathbb{E}(|Z|^2) + 2\operatorname{Re}\mathbb{E}(\bar{Y}Z))(Y + Z)$$

with w smooth enough. Scattering for the perturbed NLS (10) remains an open problem.

Nevertheless, one can prove scattering properties for (1).

4.1 Scattering for the defocusing equation

We now prove scattering in \mathbb{R}^3 .

We use Morawetz estimates in the spirit of [18] and [29]. We mention [12] about scattering for a system of Schrödinger equations.

We follow the proof for decay estimates and scattering in [28] from page 67 and onward. Because the computation for the linear part of the equation is the same up to constants, we will not insist on it and focus on the main difference, which is the non linearity.

Proposition 4.1. *The equation (1) scatters in the sense that for all initial datum X_0 in $L^2(\Omega, H^1(\mathbb{R}^3))$ there exists $X_{\pm\infty} \in L^2(\Omega, H^1(\mathbb{R}^3))$ such that*

$$\|X(t) - S(t)X_{\pm\infty}\|_{H^1(\mathbb{R}^3)} \rightarrow 0$$

when t goes to $\pm\infty$. By $X(t)$ we denote the solution of (1) with initial datum X_0 and by $S(t)$ the flow of the linear equation $\partial_t Z = -\Delta Z$.

We start with decay estimates.

Lemma 4.2. *With the notations of the previous proposition we have that $X(t)$ belongs to $L^4(\mathbb{R} \times \mathbb{R}^3, L^2(\Omega))$. In other terms, the quantity*

$$\int_{\mathbb{R}} dt \int_{\mathbb{R}^3} dx \mathbb{E}(|X(t, x)|^2)^2$$

is finite.

Proof. We start from the fact that X satisfies a conservation law written : for all $j = 0, 1, 2, 3$,

$$\partial_t T_{j0} = \sum_{k=1}^3 \partial_{x_k} T_{jk}$$

with $T_{00} = \mathbb{E}(|X|^2)$, $T_{0j} = T_{j0} = -2\operatorname{Im}\mathbb{E}(\bar{X}\partial_{x_j}X)$ for $j > 0$ and for $j, k > 0$,

$$T_{jk} = 2\operatorname{Re}(\mathbb{E}(\partial_{x_j}X\bar{\partial_{x_k}X})) - \frac{1}{2}\delta_j^k \Delta(\mathbb{E}(|X|^2)) + \delta_j^k \mathbb{E}(|X|^2)^2.$$

Indeed, for $j = 0$, we have

$$\partial_t T_{00} = 2\operatorname{Re}\mathbb{E}(\partial_t X\bar{X}) = 2\operatorname{Im}\mathbb{E}(i\partial_t X\bar{X}) = -2\operatorname{Im}\mathbb{E}(\Delta X\bar{X}) + 2\operatorname{Im}\mathbb{E}(\mathbb{E}(|X|^2)|X|^2)$$

Because of the imaginary part, the second term is 0. Besides, we have

$$\partial_{x_k} T_{0k} = -2\text{Im}\mathbb{E}(|\partial_{x_k} X|^2) - 2\text{Im}\mathbb{E}(\bar{X}\partial_{x_k}^2 X).$$

Because of the imaginary part the first term is 0 and summing over k yields

$$\partial_t T_{00} = \sum_{k=1}^3 \partial_{x_k} T_{0k}.$$

For $j > 0$, we have

$$\partial_t T_{j0} = -2\text{Im}\mathbb{E}(\partial_t \bar{X} \partial_{x_j} X) - 2\text{Im}\mathbb{E}(\bar{X} \partial_t \partial_{x_j} X) = -2\text{Re}\mathbb{E}(i \partial_t \bar{X} \partial_{x_j} X) + 2\text{Re}\mathbb{E}(\bar{X} \partial_{x_j} i \partial_t X).$$

As X solves (1), we get

$$\partial_t T_{j0} = 2\text{Re}\mathbb{E}(\bar{\Delta} \bar{X} \partial_{x_j} X) - 2\text{Re}\mathbb{E}(\bar{X} \partial_{x_j} \Delta X) - 2\text{Re}\mathbb{E}(\overline{\mathbb{E}(|X|^2)} \bar{X} \partial_{x_j} X) + 2\text{Re}\mathbb{E}(\bar{X} \partial_{x_j} (\mathbb{E}(|X|^2) X)).$$

For the same reasons as in the deterministic case, we have for the terms involving only the linear part of the equation,

$$2\text{Re}\mathbb{E}(\bar{\Delta} \bar{X} \partial_{x_j} X) - 2\text{Re}\mathbb{E}(\bar{X} \partial_{x_j} \Delta X) = \sum_{k=1}^3 \partial_{x_k} \left(2\text{Re}(\mathbb{E}(\partial_{x_j} X \overline{\partial_{x_k} X})) - \frac{1}{2} \delta_j^k \Delta (\mathbb{E}(|X|^2)) \right).$$

For the term involving the non-linearity, we have that

$$-2\text{Re}\mathbb{E}(\overline{\mathbb{E}(|X|^2)} \bar{X} \partial_{x_j} X) + 2\text{Re}\mathbb{E}(\bar{X} \partial_{x_j} (\mathbb{E}(|X|^2) X)) = 4\mathbb{E}(|X|^2) \text{Re}\mathbb{E}((\partial_{x_j} X) \bar{X})$$

and

$$\partial_{x_k} \delta_j^k \mathbb{E}(|X|^2)^2 = \delta_j^k 4\mathbb{E}(|X|^2) \text{Re}\mathbb{E}((\partial_{x_j} \bar{X}) X).$$

Summing over k yields

$$\partial_t T_{j0} = \sum_{k=1}^3 \partial_{x_k} T_{jk}.$$

Thanks to this structure, we repeat the usual computation to get

$$\partial_t \int_{\mathbb{R}^3 \times \Omega} \sum_j \frac{x_j}{|x|} \text{Im}(\bar{X} \partial_{x_j} X) dx = \int_{\mathbb{R}^3 \times \Omega} \frac{|\nabla_0 X(x)|^2}{|x|} dx + \int_{\mathbb{R}^3} \frac{\mathbb{E}(|X|^2)^2}{|x|} dx$$

where ∇_y is the angular part of the gradient centred in y and thus ∇_0 is merely the angular gradient. We get the Morawetz estimate :

$$\int_{\mathbb{R} \times \mathbb{R}^3} \frac{\mathbb{E}(|X|^2)^2}{|x|} dx dt \leq \sup_{t \in \mathbb{R}} \|X(t)\|_{H^1(\mathbb{R}^3)}^2 < \infty.$$

Translating the last equality by y , we get

$$\begin{aligned} \partial_t \int_{\mathbb{R}^3 \times \Omega} \sum_j \frac{x_j - y_j}{|x - y|} \text{Im}(\bar{X}(x) \partial_{x_j} X(x)) dx &= \int_{\mathbb{R}^3 \times \Omega} \frac{|\nabla_y X(x)|^2}{|x - y|} dx + \\ &\quad \int_{\mathbb{R}^3} \frac{\mathbb{E}(|X(x)|^2)^2}{|x - y|} dx + \pi \mathbb{E}(|X(y)|^2). \end{aligned}$$

Finally, multiplying by $\mathbb{E}(|X(y)|^2)$ and integrating over y , we get

$$\partial_t \int_{\mathbb{R}^3 \times \mathbb{R}^3} \sum_j \frac{x_j - y_j}{|x - y|} \mathbb{E}(|X(y)|^2) |\operatorname{Im} \mathbb{E}(\bar{X}(x) \partial_{x_j} X(x))| dx dy = I + II + III + IV$$

with

$$\begin{aligned} I &= \int_{\mathbb{R}^3 \times \mathbb{R}^3} \mathbb{E}(|X(y)|^2) \frac{|\nabla_y X(x)|^2}{|x - y|} dx dy \\ II &= \int_{\mathbb{R}^3 \times \mathbb{R}^3} \mathbb{E}(|X(y)|^2) \frac{\mathbb{E}(|X(x)|^2)^2}{|x - y|} dx dy \\ III &= \pi \int_{\mathbb{R}^3} \mathbb{E}(|X(y)|^2)^2 dy \\ IV &= \int_{\mathbb{R}^3 \times \mathbb{R}^3} \sum_j \frac{x_j - y_j}{|x - y|} \partial_t (\mathbb{E}(|X(y)|^2)) |\operatorname{Im} \mathbb{E}(\bar{X}(x) \partial_{x_j} X(x))| dx dy. \end{aligned}$$

The terms I and II are non negative. The term III is the one we want to estimate. For the same structural reasons as in the deterministic case, the term IV is controlled by I . Hence, we get that

$$III \leq \partial_t \int_{\mathbb{R}^3 \times \mathbb{R}^3} \sum_j \frac{x_j - y_j}{|x - y|} \mathbb{E}(|X(y)|^2) |\operatorname{Im} \mathbb{E}(\bar{X}(x) \partial_{x_j} X(x))| dx dy$$

and we get the interaction Morawetz estimate

$$\int_{\mathbb{R} \times \mathbb{R}^3} \mathbb{E}(|X|^2)^2 dx dt \leq \sup_{t \in \mathbb{R}} \|X(t)\|_{H^1(\mathbb{R}^3)}^4 < \infty$$

which concludes the proof. \square

Let I be an interval of \mathbb{R} . We call \mathcal{L}_I the space

$$\mathcal{L}_I = L^{10}(I, L^{10}(\mathbb{R}^3)) \cap L^{10/3}(I, W^{1,10/3}(\mathbb{R}^3)).$$

Lemma 4.3. *With the notations of Proposition 4.1, we have $X \in L^2(\Omega, \mathcal{L}_{\mathbb{R}})$.*

Proof. Let $I = [t_1, t_2]$. For all $t \in T$, the Duhamel formula of (1) writes

$$X(t) = S(t - t_1)X(t_1) - i \int_{t_1}^t S(t - \tau) (\mathbb{E}(|X(\tau)|^2) X(\tau)) d\tau.$$

We have that $W^{1,30/13}(\mathbb{R}^3)$ is embedded in $L^{10}(\mathbb{R}^3)$ by Sobolev's embedding, and $(10, \frac{30}{13})$ and $(\frac{10}{3}, \frac{10}{3})$ are admissible for the Schrödinger dispersion in dimension 3, since

$$\frac{2}{10} + \frac{3}{30/13} = \frac{15}{10} = \frac{3}{2} \text{ and } \frac{2}{10/3} + \frac{3}{10/3} = \frac{3}{2}.$$

Besides, $\frac{10}{7}$ is the conjugate of $\frac{10}{3}$ hence, thanks to Strichartz estimates and a TT^* argument

$$\|X\|_{\mathcal{L}_I} \leq C \|X(t_1)\|_{H^1} + C \|\mathbb{E}(|X|^2) X\|_{L^{10/7}(I, W^{1,10/7}(\mathbb{R}^3))}.$$

We use the fact that $\frac{10}{7} \leq 2$ to apply Minkowski inequality and get

$$\|X\|_{L^2(\Omega, \mathcal{L}_I)} \leq C \|X(t_1)\|_{L^2(\Omega, H^1)} + C \|D(\mathbb{E}(|X|^2) X)\|_{L^{10/7}(I, L^{10/7}(\mathbb{R}^3, L^2(\Omega)))}.$$

Distributing the derivative, we get

$$\|D(\mathbb{E}(|X|^2)X)\|_{L^2(\Omega)} \leq \|DX\|_{L^2(\Omega)} \|X\|_{L^2(\Omega)}^2.$$

Using Hölder's inequality with $2\frac{1}{5} + \frac{3}{10} = \frac{7}{10}$, we get

$$\|D(\mathbb{E}(|X|^2)X)\|_{L^{10/7}(I, L^{10/7}(\mathbb{R}^3, L^2(\Omega)))} \leq \|DX\|_{L^{10/3}(I \times \mathbb{R}^3, L^2(\Omega))} \|X\|_{L^5(I \times \mathbb{R}^3, L^2(\Omega))}^2.$$

Using again Minkowski's inequality as $\frac{10}{3} \geq 2$, we get

$$\|DX\|_{L^{10/3}(I \times \mathbb{R}^3, L^2(\Omega))} \leq \|X\|_{L^2(\Omega, L^{10/3}(I, W^{1,10/3}(\mathbb{R}^3)))} \leq \|X\|_{L^2(\Omega, \mathcal{L}_I)}.$$

Using that 5 lies between 4 and 10, we get

$$\|X\|_{L^5(I \times \mathbb{R}^3, L^2(\Omega))} \leq \|X\|_{L^4(I \times \mathbb{R}^3, L^2(\Omega))}^{2/3} \|X\|_{L^{10}(I \times \mathbb{R}^3, L^2(\Omega))}^{1/3}.$$

Using once more Minkowski's inequality and the definition of \mathcal{L}_I , we have

$$\|X\|_{L^{10}(I \times \mathbb{R}^3, L^2(\Omega))} \leq \|X\|_{L^2(\Omega, \mathcal{L}_I)}.$$

Besides, we use that thanks to the conservation of the energy the quantity $\|X(t_1)\|_{L^2(\Omega, H^1)}$ is bounded uniformly in t_1 by a quantity \mathcal{E}_0 .

Summing up, we get

$$\|X\|_{L^2(\Omega, \mathcal{L}_I)} \leq C\mathcal{E}_0 + C\|X\|_{L^2(\Omega, \mathcal{L}_I)}^{5/3} \|X\|_{L^4(I \times \mathbb{R}^3, L^2(\Omega))}^{4/3}.$$

Let $\varepsilon = \mathcal{E}_0^{-1/2}(2C)^{-5/2}$. As, by Lemma 4.2

$$\|X\|_{L^4(\mathbb{R} \times \mathbb{R}^3, L^2(\Omega))} = \left(\int_{\mathbb{R}} dt \int_{\mathbb{R}^3} dx \mathbb{E}(|X(t, x)|^2)^2 \right)^{1/4}$$

is finite, there exist a finite family of intervals $(I_j)_{1 \leq j \leq r}$ such that

$$\bigcup_{i=1}^r I_j = \mathbb{R} \text{ and for all } j, \|X\|_{L^4(I_j \times \mathbb{R}^3, L^2(\Omega))} \leq \varepsilon.$$

Therefore, for all j , we get

$$\|X\|_{L^2(\Omega, \mathcal{L}_{I_j})} \leq C\mathcal{E}_0 + C\|X\|_{L^2(\Omega, \mathcal{L}_{I_j})}^{5/3} \varepsilon^{4/3}.$$

This choice of ε implies $\|X\|_{L^2(\Omega, \mathcal{L}_{I_j})} \leq 2C\mathcal{E}_0$. Summing over j yields

$$\|X\|_{L^2(\Omega, \mathcal{L}_{\mathbb{R}})} \lesssim \mathcal{E}_0 < \infty$$

hence the result. □

We describe $X_{\pm\infty}$.

Lemma 4.4. *Let*

$$X_{\pm\infty} = X_0 - i \int_0^{\pm\infty} S(-\tau) \mathbb{E}(|X(\tau)|^2) X(\tau) d\tau.$$

The maps $X_{\pm\infty}$ belong to $L^2(\Omega, H^1(\mathbb{R}^3))$.

Proof. First, $X_0 \in L^2(\Omega, H^1(\mathbb{R}^3))$. Then, thanks to Strichartz estimates and a T^* argument, we get

$$\left\| \int_0^{\pm\infty} S(-\tau) \mathbb{E}(|X(\tau)|^2) X(\tau) d\tau \right\|_{L^2(\Omega, H^1(\mathbb{R}^3))} \leq C \|D(\mathbb{E}(|X|^2)X)\|_{L^2(\Omega, L^{10/7}(\mathbb{R} \times \mathbb{R}^3))}.$$

With the same computation as previously, we get

$$\left\| \int_0^{\pm\infty} S(-\tau) \mathbb{E}(|X(\tau)|^2) X(\tau) d\tau \right\|_{L^2(\Omega, H^1(\mathbb{R}^3))} \leq C \|X\|_{L^2(\Omega, L^5(\mathbb{R} \times \mathbb{R}^3))}^2 \|X\|_{L^2(\Omega, L^{10/3}(\mathbb{R}, W^{1,10/3}(\mathbb{R}^3)))}$$

which is finite by interpolation. \square

Proof of Proposition 4.1. We focus on $+\infty$. We have

$$\begin{aligned} \|X(t) - S(t)X_{+\infty}\|_{H^1(\mathbb{R}^3)} &= \left\| \int_t^\infty S(t-\tau) \mathbb{E}(|X(\tau)|^2) X(\tau) d\tau \right\|_{H^1(\mathbb{R}^3)} \\ &\leq C \|1_{\tau \geq t} X\|_{L^2(\Omega, L^5(\mathbb{R} \times \mathbb{R}^3))}^2 \|1_{\tau \geq t} X\|_{L^2(\Omega, L^{10/3}(\mathbb{R}, W^{1,10/3}(\mathbb{R}^3)))} \end{aligned}$$

which goes to 0 as t goes to ∞ . We use the dominated convergence theorem to handle the $L^2(\Omega)$ norm. \square

4.2 Lack of localised equilibrium

Proposition 4.5. *Let Y be a solution of (1) whose law is invariant in time. Assume that $Y(t=0)$ belongs to $L^2(\Omega, H^1(\mathbb{R}^3))$. Then $Y = 0$.*

Proof. Indeed, if Y is in $L^2(\Omega, H^1(\mathbb{R}^3))$ then thanks to lemma 4.3, Y belongs to $L^2(\Omega, L^{10}(\mathbb{R} \times \mathbb{R}^3))$ which is continuously embedded in $L^{10}(\mathbb{R} \times \mathbb{R}^3, L^2(\Omega))$. We have

$$\|Y\|_{L^{10}(\mathbb{R} \times \mathbb{R}^3, L^2(\Omega))}^{10} = \int_{\mathbb{R}} dt \int_{\mathbb{R}^3} dx \mathbb{E}(|Y(t, x)|^2)^5.$$

Because the law of Y does not depend on time, we have that $\mathbb{E}(|Y(t, x)|^2)^5$ is a map $\varphi(x)$ which does not depend on time. Hence $\int_{\mathbb{R}^3} dx \mathbb{E}(|Y(t, x)|^2)^5$ is a constant and thus, for it to be integrable, it has to be 0, which ensures that $Y = 0$. \square

5 On the focusing case

Up to now, we have only considered the defocusing case but we can now consider the focusing equation :

$$i\partial_t X = -\Delta X - \mathbb{E}(|X|^2)X \quad (16)$$

in \mathbb{R}^d , $d \leq 3$.

First of all, this equation is locally well-posed for initial data taken in $H^1(\mathbb{R}^d)$, $d \leq 3$.

Besides, we remark that (16) has stationary solutions. Let Q be a stationary solution of $i\partial_t u = -\Delta u - |u|^2 u$ and X be a random variable such that the probability that $X = Q$ is 1. Then X is a stationary solution of (16).

We prove the existence of blow-up solutions for the focusing equation.

We proceed with a viriel method. We prove that

$$V(t) = \int_{\Omega \times \mathbb{R}^d} |x|^2 |X|^2 \quad (17)$$

is well-defined on $[0, T]$ as long as the solution X of (16) is well posed on $[0, T]$.

Lemma 5.1. *Let φ be a non negative C^1 function on \mathbb{R}^d with compact support. We have*

$$\partial_t \left(\int_{\Omega \times \mathbb{R}^d} \varphi(x) |X|^2 \right) = 2 \operatorname{Im} \int_{\Omega \times \mathbb{R}^d} \nabla \varphi \bar{X} \nabla X.$$

Proof. The computation is the same as in the deterministic case, which yields

$$\partial_t \left(\int_{\Omega \times \mathbb{R}^d} \varphi(x) |X|^2 \right) = 2 \operatorname{Im} \int_{\Omega \times \mathbb{R}^d} \varphi(x) \bar{X} (-\Delta X + \mathbb{E}(|X|^2) X).$$

We have $\varphi(x) \bar{X} \mathbb{E}(|X|^2) X \in \mathbb{R}$ thus we keep only

$$2 \operatorname{Im} \int_{\Omega \times \mathbb{R}^d} \varphi(x) \bar{X} (-\Delta X)$$

and with an integration by parts we get

$$2 \operatorname{Im} \int_{\Omega \times \mathbb{R}^d} \nabla(\varphi(x) \bar{X}) \nabla X$$

and by developing the gradient and seeing that $\varphi |\nabla X|^2 \in \mathbb{R}$, we get the result. \square

Let φ the specific function such that

$$\varphi(x) = \begin{cases} |x|^2 & \text{if } |x| \leq 1 \\ e^{1-1/(|x|-2)^2} & \text{if } |x| \in [1, 2] \\ 0 & \text{otherwise.} \end{cases}$$

We have $\varphi \in C^1$ with compact support and there exists C such that for all $x \in \mathbb{R}^d$, $|\nabla \varphi(x)|^2 \leq C \varphi(x)$.

Lemma 5.2. *Assuming that $V(t = 0)$ is well-defined, the Viriel $V(t)$ is well-defined on $[0, T]$ as long as the solution X of (16) is well posed on $[0, T]$.*

Proof. For all $R > 0$ let $\varphi_R(x) = R^2 \varphi(\frac{x}{R})$. We have

$$\int_{|x| \leq R} |x|^2 |X|^2 \leq \int \varphi_R(x) |X|^2.$$

We apply the last lemma to get

$$\partial_t \left(\int \varphi_R(x) |X|^2 \right) = 2 \operatorname{Im} \int_{\Omega \times \mathbb{R}^d} \nabla \varphi_R \bar{X} \nabla X.$$

We apply Cauchy-Schwartz inequality to get

$$\left| \partial_t \left(\int \varphi_R(x) |X|^2 \right) \right| \leq 2 \|\nabla \varphi_R \bar{X}\| \|X(t)\|_{H^1}.$$

We use that $|\nabla \varphi_R(x)|^2 = |R \nabla \varphi(\frac{x}{R})|^2 \leq C R^2 \varphi(\frac{x}{R}) = \varphi_R(x)$ to get

$$\left| \partial_t \left(\int \varphi_R(x) |X|^2 \right) \right| \leq C \left(\int \varphi_R(x) |X|^2 \right)^{1/2} \|X(t)\|_{H^1}$$

From which we deduce

$$\left(\int \varphi_R(x) |X|^2 \right)^{1/2} \leq V(t = 0)^{1/2} + C \int_0^T \|X(\tau)\|_{H^1} d\tau.$$

As the right hand side is bounded uniformly in R , we get the result. \square

We compute the second derivative of V .

Lemma 5.3. *We have, where V is defined*

$$\partial_t^2 V(t) \leq 16\mathcal{E}(X_0).$$

Proof. We have, thanks to Lemma 5.1

$$\partial_t V = 4\text{Im} \int_{\Omega \times \mathbb{R}^d} x \nabla X \bar{X}.$$

We differentiate it a second time to get

$$\partial_t^2 V = I + II$$

with

$$I = 4\text{Re} \int_{\Omega \times \mathbb{R}^d} x (\overline{i\partial_t X} \nabla X) \text{ and } II = -4\text{Re} \int_{\Omega \times \mathbb{R}^d} x (\bar{X} \nabla (i\partial_t X)).$$

By integration by parts, we get that II is given by

$$4\text{Re} \int \bar{X} (i\partial_t X) + 4\text{Re} \int x \nabla \bar{X} (i\partial_t X)$$

and thus, by replacing $i\partial_t X$ by its value,

$$\partial_t^2 V(t) = 4d \int_{\Omega \times \mathbb{R}^d} \bar{X} (-\Delta) X + 4d \int_{\mathbb{R}^d} \mathbb{E}(|X|^2)^2 + 2I.$$

We compute I . By replacing $i\partial_t X$ by its value, we get

$$I = I.1 + I.2$$

with

$$I.1 = 4\text{Re} \int_{\Omega \times \mathbb{R}^d} x \nabla X (-\Delta \bar{X}) \text{ and } I.2 = 4\text{Re} \int_{\Omega \times \mathbb{R}^d} x \nabla X (-\mathbb{E}(|X|^2) \bar{X})$$

The computation for $I.1$ is the same as in the deterministic case, and we get

$$I.1 = (2d - 4) \int_{\Omega \times \mathbb{R}^d} X \Delta \bar{X}.$$

The computation for $I.2$ requires to take into account the probability. We replace the gradient by partial derivatives to get

$$I.2 = -4\text{Re} \sum_j \int_{\Omega \times \mathbb{R}^d} x_j \mathbb{E}(|X|^2) \bar{X} \partial_j X$$

where $\partial_j = \partial_{x_j}$. We replace the integral in Ω by the expectation \mathbb{E} to get

$$I.2 = -4\text{Re} \sum_j \int_{\mathbb{R}^d} x_j \mathbb{E}(|X|^2) \mathbb{E}(\bar{X} \partial_j X).$$

We remark that $\partial_j \mathbb{E}(|X|^2)^2 = 4\text{Re} \mathbb{E}(|X|^2) \mathbb{E}(\bar{X} \partial_j X)$ such that

$$I.2 = - \sum_j \int_{\mathbb{R}^d} x_j \partial_j \mathbb{E}(|X|^2)^2$$

and by integration by parts

$$I.2 = d \int_{\mathbb{R}^d} \mathbb{E}(|X|^2)^2.$$

Summing up, we get

$$\partial_t^2 V(t) = 8 \int_{\Omega \times \mathbb{R}^d} \bar{X}(-\Delta)X - 2d \int_{\mathbb{R}^d} \mathbb{E}(|X|^2)^2$$

and for $d \geq 2$,

$$\partial_t^2 V \leq 16\mathcal{E}(X(t)) = 16\mathcal{E}(X_0).$$

□

Proposition 5.4. *If $X_0 \in L^2(\Omega, H^1(\mathbb{R}^d))$ is such that $V(t = 0)$ is finite and $\mathcal{E}(X_0) < 0$, then the solution of (16) blows up at finite time.*

6 Incidence at the operator level

6.1 Incidence at the operator level on the sphere and torus

In this section, we prove the global well-posedness of (2) on the sphere and torus.

Let $M \in \{\mathbb{S}^2, \mathbb{S}^3, \mathbb{T}^2, \mathbb{T}^3\}$.

6.1.1 Uniqueness of laws

In this subsection we prove that two solutions of (1) whose initial data have the same law have also the same law. For this setting, it is relevant to use Subsection 2.4. Nevertheless, since the following technique is easier to expose in this setting rather than for the perturbed equation and since we require it for the perturbed equation, we choose to present it here.

Lemma 6.1. *Let $X(t)$ be a solution of (1) with initial datum X_0 defined on the probability space (Ω, \mathcal{F}, P) and belonging to $L^2(\Omega, H^1(M))$. Let $(\omega_1, \omega_2) \in \Omega^2$. If $X_0(\omega_1) = X_0(\omega_2)$, then at all times t , $X(t, \omega_1) = X(t, \omega_2)$.*

Proof. Let $\varphi(t, x) = \mathbb{E}(|X(t, x)|^2)$. Both $X(t, \omega_1)$ and $X(t, \omega_2)$ are solutions of

$$i\partial_t u = -\Delta u + \varphi(t, x)u$$

with the same initial datum $u_0 = X_0(\omega_1) = X_0(\omega_2)$. In view of the previous sections, this ensures that $X(t, \omega_1) = X(t, \omega_2)$. □

Definition 6.2. Given an initial datum X_0 defined on the probability space (Ω, \mathcal{F}, P) and belonging to $L^2(\Omega, H^1(M))$, let \sim_P be the equivalence relation on Ω defined as

$$\omega_1 \sim_P \omega_2 \Leftrightarrow X_0(\omega_1) = X_0(\omega_2).$$

Let $(\Omega', \mathcal{F}', P')$ be the probability space (Ω, \mathcal{F}, P) quotiented by \sim_P , that is

$$\begin{aligned} \Omega' &= \{cl(\omega) \mid \omega \in \Omega\}, \\ \mathcal{F}' &= \{cl(X_0^{-1}(A)) \mid A \text{ measurable in } H^1(M)\}, \\ \forall C \in \mathcal{F}', P'(C) &= P\left(\bigcup_{c \in C} c\right) \end{aligned}$$

where

$$\begin{aligned} cl(\omega) &= \{\omega' \in \Omega \mid \omega' \sim_P \omega\} \\ cl(A) &= \{cl(\omega) \mid \omega \in A\}. \end{aligned}$$

Finally, let $X'(t)$ be the random variable defined on $(\Omega', \mathcal{F}', P')$ and belonging to $L^2(\Omega', H^1(M))$ as $X'(t)(cl(\omega)) = X(t)(\omega)$.

Remark 6.1. *The measure P' is well-defined on \mathcal{F}' and*

$$P'(cl(X_0^{-1}(A))) = P(X_0^{-1}(A)).$$

Indeed, if $\omega \in X_0^{-1}(A)$, then $cl(\omega) \subseteq X_0^{-1}(A)$.

The random variable $X'(t)$ is defined without ambiguity thanks to Lemma 6.1. It belongs to $L^2(\Omega', H^1(M))$ since

$$\mathbb{E}(\|X'(t)\|_{H^1}^2) = \int_{\mathbb{R}^+} P'(X'(t)^{-1}(B_{H^1}(0, \sqrt{\lambda})^c)) d\lambda$$

where c stands for the complementary set. Given the definition of P' , this yields

$$\mathbb{E}(\|X'(t)\|_{H^1}^2) = \int_{\mathbb{R}^+} P(X(t)^{-1}(B_{H^1}(0, \sqrt{\lambda})^c)) d\lambda = \mathbb{E}(\|X(t)\|_{H^1}^2) < \infty.$$

Lemma 6.3. *The law of $X'(t)$ is the same as the law of $X(t)$.*

Proof. This is due that for all measurable A set in $H^1(M)$, we have

$$X'(t)^{-1}(A) = cl(X(t)^{-1}(A))$$

and due to the definition of P . Note that $X(t)^{-1}(A)$ is measurable in Ω and $X'(t)^{-1}(A)$ measurable in Ω' because the flow of (1) is continuous. \square

Lemma 6.4. *Let X_1 and X_2 be solutions to (1) with initial datum $X_{1,0} \in \mathcal{L}^2(\Omega_1, H^1(M))$ and $X_{2,0} \in \mathcal{L}^2(\Omega_2, H^1(M))$ which have the same law. Then, for all t , $X_1(t)$ and $X_2(t)$ have the same law.*

Proof. Thanks to Lemma 6.3 and using the same notations, we can consider the random variables X'_1 and X'_2 instead of X_1 and X_2 . Let φ be the map from Ω'_1 to Ω'_2 defined as

$$\varphi(X_{0,1}^{-1}(\{u_0\})) = X_{0,2}^{-1}(\{u_0\})$$

for all $u_0 \in H^1(M)$.

By construction, $X'_{1,0} = X'_{2,0} \circ \varphi$ and P'_2 is the image measure of P'_1 under φ .

By uniqueness of the flow of (1), $X'_2(t) \circ \varphi = X'_1(t)$ and since φ preserves the measure the law of $X'_2(t)$ is the same as the one of $X'_1(t)$. Therefore, thanks to Lemma 6.3, $X_1(t)$ and $X_2(t)$ have the same law. \square

6.1.2 Gaussian variables

In this subsection, we prove that if X_0 is a Gaussian variable, then so is $X(t)$ the solution of (1) with initial datum X_0 . What is more, we prove that if γ is a solution of (2) then there exists a Gaussian variable with covariance γ that is a solution of (1).

Lemma 6.5. *Let X_0 be a Gaussian process of covariance γ_0 with $\text{Tr}((1 - \Delta)\gamma_0) < \infty$. Let $X(t)$ be the solution of (1) with initial datum X_0 then $X(t)$ is a Gaussian process.*

Proof. Write $\varphi(t, x) = \mathbb{E}(|X(t, x)|^2)$. By Propositions 2.1, and 2.2 one gets that the equation

$$i\partial_t u = -\Delta u + \varphi u$$

is well-posed in H^1 . Let $U(t)$ be the flow of this equation, it is linear and continuous on H^1 . Let $\lambda \in H^{-1}$. Since by uniqueness of the flow we have $X(t) = U(t)X_0$,

$$\mathbb{E}(e^{i\langle \lambda, X(t) \rangle}) = \mathbb{E}(e^{i\langle U^*(t)\lambda, X_0 \rangle}) = e^{-\langle U^*(t)\lambda, \gamma_0 U^*(t)\lambda \rangle} = e^{-\langle \lambda, U(t)\gamma_0 U^*(t)\lambda \rangle}.$$

Since $U(t)\gamma_0 U^*(t)$ is a positive operator, we get that $X(t)$ is a Gaussian process of covariance $U(t)\gamma_0 U^*(t)$. \square

Definition 6.6. Let Σ be the set of non negative operators γ_0 such that $\text{Tr}((1 - \Delta)\gamma_0)$ is finite on M . Let d be the distance on this set defined as

$$d(\gamma_1, \gamma_2) = d_2(\nu_1, \nu_2)$$

where d_2 is the Wasserstein distance defined in Remark 2.3 and ν_i is the law of the Gaussian process with covariance γ_i .

Remark 6.2. *The Wasserstein distance may also be defined as*

$$d_2(\nu_1, \nu_2) = \inf_{X_i \sim \gamma_i} \|X_1 - X_2\|_{L^2(\Omega, H^1(M))}$$

where \sim stands for “is a Gaussian random field of covariance”.

By Σ , we now denote the metric space (Σ, d) .

Lemma 6.7. *Assume that $\gamma \in C(\mathbb{R}, \Sigma)$ is a solution to (2). Then there exists a probability space Ω and a Gaussian variable $X \in C(\mathbb{R}, L^2(\Omega, H^1(M)))$ solution of (1) and of covariance γ .*

Proof. Let $\varphi = \rho_\gamma$ and let X be the solution to

$$i\partial_t X = (-\Delta + \varphi)X$$

with initial datum X_0 a Gaussian variable with covariance γ_0 . Its covariance γ_X is the unique solution to the linear equation

$$i\partial_t \gamma_X = [-\Delta + \varphi, \gamma_X]$$

with initial datum $\gamma_{X_0} = \gamma(t = 0)$.

Indeed, let $U(t)$ the flow of the linear equation on $u : i\partial_t u = (-\Delta + \varphi)u$. The map U is invertible. Hence, since,

$$i\partial_t (U(t)^* \gamma_X U(t)) = 0$$

the equation $i\partial_t \gamma_X = [-\Delta + \varphi, \gamma_X]$ has a unique solution.

Since γ is also a solution to $i\partial_t \gamma_X = [-\Delta + \varphi, \gamma_X]$ with initial datum $\gamma(t = 0)$ we get that $\gamma_X = \gamma$, thus $\mathbb{E}(|X|^2) = \rho$, which ensures that X is a sol to (1). \square

6.1.3 Global well-posedness

Corollary 6.8 (of Proposition 2.5). *Let Ψ be the map from Σ to $C(\mathbb{R}, \Sigma)$ such that $\Psi(t)(\gamma_0) = \gamma_{X(t)}$ where $X(t)$ is the solution to (1) with initial datum X_0 the Gaussian random process with covariance operator γ_0 . The map Ψ is well-defined, it defines a solution to (2) and it is continuous for the distance d . Besides, $\Psi(t)\gamma_0$ is the unique solution to (2) with initial datum γ_0 .*

Proof. Because of Proposition 1.2 we get that an initial datum $\gamma_0 \in \Sigma$ at the operator level such that $\text{Tr}((1 - \Delta)\gamma_0)$ is finite gives an initial datum at the level of random variables X_0 belonging to $L^2(\Omega, H^1(M))$. Thanks to Proposition 2.5 we get a solution X of (1), and thanks to Proposition 1.1, we get a solution γ to the equation (2). We remark that thanks to Lemma 6.4, one can take any Gaussian X_0 with covariance γ_0 as the law of $X(t)$ depends only on the law of X_0 . Hence $\Psi(t)$ is well-defined. It is continuous in time and in the initial datum for the following reason : the distance between $\gamma_{X_1(t)}$ and $\gamma_{X_2(t)}$ is controlled by the norm of $X_1(t) - X_2(t)$. Indeed, X_i is a Gaussian process by Lemma 6.5, therefore

$$d(\gamma_1(t_1), \gamma_2(t_2)) \leq \|X_1(t_1) - X_2(t_2)\|_{H^1(M)}$$

where $\gamma_i(t)$ is equal to $\gamma_{X_i(t)}$.

The continuity of the solution $X(t)$ in both time and initial datum gives the result. Indeed, take any Gaussian process X_i with covariance γ_i and any couple of times t_1, t_2 , we have

$$d(\gamma_1(t_1), \gamma_1(t_2)) \leq \|X_1(t_1) - X_1(t_2)\|_{L^2(\Omega, H^1)} \leq C(X_1)|t_1 - t_2|^\alpha$$

for some $\alpha > 0$ and $C(X_1) = C(\gamma_1)$ is a constant depending only on γ_1 . What is more,

$$d(\gamma_1(t_1), \gamma_2(t_1)) \leq \|X_1(t_1) - X_2(t_1)\|_{L^2(\Omega, H^1)} \leq C(t_1)\|X_1 - X_2\|_{L^2(\Omega, H^1(M))}$$

and by taking the infimum over the couples (X_1, X_2) we get the result.

For the uniqueness of the solution, let γ_1 and γ_2 be two solutions of (2) with the initial datum γ_0 . For $i = 1, 2$, there exists $X_i(t)$ a solution of (1) which is a Gaussian variable of covariance γ_i . For $i = 1, 2$, $X_i(t = 0)$ is a Gaussian variable of covariance γ_0 . Hence $X_1(t = 0)$ and $X_2(t = 0)$ have the same law. Therefore $X_1(t)$ and $X_2(t)$ too, which ensures that $\gamma_1 = \gamma_2$ and hence the uniqueness of the solution of (2). \square

Remark 6.3. *One could rewrite the corollary 6.8 as : the equation (2) is globally well-posed in $C(\mathbb{R}, \Sigma)$.*

6.2 Global well-posedness on the Euclidean space

Let f be a bounded function on \mathbb{R}^d such that $\langle k \rangle f(k) \in L^2(\mathbb{R}^d)$ and let Y_f be the equilibrium corresponding to f that is

$$Y_f(t, x) = \int f(k) e^{i(m+k^2)t} e^{ikx} dW_k$$

with $m = \int |f(k)|^2 dk$.

This random variable defines an equilibrium for (1) and the operator $\gamma_f = \gamma_{Y_f}$ is a stationary solution for (2). Indeed, γ_f is the Fourier multiplier by $|f(k)|^2$. Hence it commutes with the Laplacian and $\rho_{\gamma_f} = m$.

In this section, we prove the global well-posedness of (2) around equilibria γ_f , that is, we prove global well-posedness of the equation

$$i\partial_t Q = [-\Delta, Q] + [\rho_{\gamma_f+Q}, \gamma_f + Q] \quad (18)$$

where Q is not necessarily non-negative but $\gamma_f + Q$ is.

Let $M \in \{\mathbb{R}^2, \mathbb{R}^3\}$.

6.2.1 Uniqueness of laws

In this subsection we prove that two solutions of (1) whose initial data have the same law have also the same law.

Lemma 6.9. *Let $X(t)$ be a solution of (1) with initial datum $X_0 = Y_0 + Z_0$ defined on the probability space (Ω, \mathcal{F}, P) and such that Y_0 has the same law as $Y_f(t = 0)$ and such that Z_0 belongs to $L^2(\Omega, H^1(M))$. Write $Y(t) = e^{-it(-\Delta+m)} Y_0$ and $X(t) = Y(t) + Z(t)$. Assume $Z \in C(\mathbb{R}, (L^2(\Omega, H^1(M))))$. Let $(\omega_1, \omega_2) \in \Omega^2$. If $(Y_0, Z_0)(\omega_1) = (Y_0, Z_0)(\omega_2)$, then at all times t , $(Y, Z)(t, \omega_1) = (Y, Z)(t, \omega_2)$.*

Proof. First, if $Y_0(\omega_1) = Y_0(\omega_2)$, then $Y(t, \omega_1) = Y(t, \omega_2)$. Let $\varphi(t, x) = \mathbb{E}(|X(t, x)|^2) - m$. Both $Z(t, \omega_1)$ and $Z(t, \omega_2)$ are solutions of

$$i\partial_t u = (m - \Delta)u + \varphi(t, x)(u + Y(t, \omega_1))$$

with the same initial datum $u_0 = Z_0(\omega_1) = Z_0(\omega_2)$. In view of the previous sections, this ensures that $Z(t, \omega_1) = Z(t, \omega_2)$. \square

Definition 6.10. Given an initial datum $X_0 = Y_0 + Z_0$ defined on the probability space (Ω, \mathcal{F}, P) and with Z_0 belonging to $L^2(\Omega, H^1(M))$, and Y_0 with the same law as $Y_f(t = 0)$. Let \sim_P be the equivalence relation on Ω defined as

$$\omega_1 \sim_P \omega_2 \Leftrightarrow (Y_0, Z_0)(\omega_1) = (Y_0, Z_0)(\omega_2).$$

Let $(\Omega', \mathcal{F}', P')$ be the probability space (Ω, \mathcal{F}, P) quotiented by \sim_P , and let $Z'(t)$ be the random variable defined on $(\Omega', \mathcal{F}', P')$ and belonging to $L^2(\Omega', H^1(M))$ as $Z'(t)(cl(\omega)) = Z(t)(\omega)$ and let $Y'(t)(cl(\omega)) = Y(t)(\omega)$

Remark 6.4. *The random variable $Z'(t)$ is defined without ambiguity thanks to Lemma 6.9. It belongs to $L^2(\Omega', H^1(M))$.*

Lemma 6.11. *The law of $X'(t)$ is the same as the law of $X(t)$.*

Lemma 6.12. *Let X_1 and X_2 be solutions to (1) written $X_i = Y_i + Z_i$ with initial datum $X_{1,0}$ and $X_{2,0}$ which have the same law. The random variables Y_i satisfy $Y_i(t) = e^{-it(m-\Delta)} Y_{0,i}$ with $Y_{0,i}$ a random variable with the same law as $Y_f(t = 0)$. Then, for all t , $X_1(t)$ and $X_2(t)$ have the same law.*

Proof. Thanks to Lemma 6.11 and using the same notations, we can consider the random variables X'_1 and X'_2 instead of X_1 and X_2 . Let φ be the map from Ω'_1 to Ω'_2 defined as

$$\varphi((Y_{0,1}, Z_{0,1})^{-1}(\{(u_0, u_1)\})) = (Y_{0,2}, Z_{0,2})^{-1}(\{(u_0, u_1)\})$$

for all $u_1 \in H^1(M)$ and u_0 in the image of $Y_f(t = 0)$.

By construction, $Z'_{1,0} = Z'_{2,0} \circ \varphi$ and P'_2 is the image measure of P'_1 under φ .

By uniqueness of the flow of (10), $Z'_2(t) \circ \varphi = Z'_1(t)$ and since φ preserves the measure the law of $X'_2(t)$ is the same as the one of $X'_1(t)$. Therefore, thanks to Lemma 6.11, $X_1(t)$ and $X_2(t)$ have the same law. \square

6.2.2 Gaussian variables

In this subsection, we prove that if X_0 is a Gaussian variable, then so is $X(t)$ the solution of (1) with initial datum X_0 . What is more, we prove that if γ is a solution of (2) then there exists a Gaussian variable with covariance γ that is a solution of (1).

Lemma 6.13. *Let X_0 be a Gaussian process of covariance γ_0 such that there exists a square root $\gamma_0^{1/2}$ of γ_0 satisfying $\text{Tr}((\gamma_0^{1/2} - \gamma_f^{1/2})^*(1 - \Delta)(\gamma_0^{1/2} - \gamma_f^{1/2})) < \infty$. Let $X(t)$ be the solution of (1) with initial datum X_0 then $X(t)$ is a Gaussian process.*

Proof. Write $\varphi(t, x) = \mathbb{E}(|X(t, x)|^2)$. We have $\varphi \in m + C(\mathbb{R}, L^2(\mathbb{R}^d)) + C(\mathbb{R}, L^1(\mathbb{R}^d))$. Write $Q_0 = \gamma_0^{1/2} - \gamma_f^{1/2}$ and $W_0 = \int e^{ikx} dW(k)$. We have

$$\text{Tr}(Q_0^*(1 - \Delta)Q_0) < \infty.$$

Hence $Q_0^*(1 - \Delta)Q_0$ can be diagonalised into $Q_0^*(1 - \Delta)Q_0 = \sum_n \alpha_n |u_n \times u_n|$ with u_n orthonormal in L^2 , $\alpha_n \geq 0$ and $\sum_n \alpha_n = \text{Tr}(Q_0^*(1 - \Delta)Q_0) < \infty$. After some manipulations of the expression, we have that

$$\|Q_0 W_0\|_{L^2(\Omega, H^1(\mathbb{R}^d))}^2 = \int \int |(1 - \Delta)^{1/2} Q_0 e^{ikx}|^2 dx dk.$$

By using the decomposition of $Q_0^*(1 - \Delta)Q_0$ we get

$$\|Q W_0\|_{L^2(\Omega, H^1(\mathbb{R}^d))}^2 = \int \sum_n \alpha_n |\hat{u}_n(k)|^2 = \sum_n \alpha_n < \infty$$

where \hat{u}_n is the Fourier transform of u_n . Hence $Z_0 = Q_0 W_0$ belongs to $L^2(\Omega, H^1(\mathbb{R}^d))$.

We derive an equation on Q assuming that X is equal to $X(t) = (Q(t) + e^{-it(m-\Delta)} \gamma_f^{1/2}) W_0 = Q(t) W_0 + Y_f$. We get

$$i\partial_t Q = (\varphi - \Delta)Q + (\varphi - m)e^{-it(m-\Delta)} \gamma_f^{1/2}.$$

Finally, write $V = e^{it(m-\Delta)} Q$, we get

$$i\partial_t V = e^{it(m-\Delta)} (\varphi - m) e^{-it(m-\Delta)} (V + \gamma_f^{1/2})$$

This equation is at least locally well-posed in $\mathcal{L}(H^1)$ for instance (this is why we require f bounded) and the solution satisfies

$$e^{-it(m-\Delta)} V W_0 \in L^2(\Omega, H^1(\mathbb{R}^d)).$$

By uniqueness of the solution of (10) we get that $X(t) = (Q(t) + e^{-it(m-\Delta)} \gamma_f^{1/2}) W_0$ first locally in time and then globally. This ensures that $X(t)$ is a Gaussian variable (of covariance $(Q(t) + e^{-it(m-\Delta)} \gamma_f^{1/2})^2$).

□

Definition 6.14. Let Σ_f be the set of non negative operators γ_0 such that there exists a square root of γ_0 , $\gamma_0^{1/2}$ such that $\text{Tr}((\gamma_0^{1/2} - \gamma_f^{1/2})^*(1 - \Delta)(\gamma_0^{1/2} - \gamma_f^{1/2}))$ is finite. Let d_f be the distance on this set defined as

$$d_f(\gamma_1, \gamma_2) = d_2(\nu_1, \nu_2)$$

where d_2 is the Wasserstein distance defined in Remark 2.3 and ν_i is the law of the Gaussian process with covariance γ_i .

Remark 6.5. This distance is well-defined. Let $W_0 = \int e^{ikx} dW(k)$ where W is defined as in the definition of Y_f . We have that $X_i = \gamma_i^{1/2} W_0$ is a Gaussian variable with covariance operator γ_i . Hence

$$d_f(\gamma_1, \gamma_2) \leq \|X_1 - X_2\|_{L^2(\Omega, H^1(M))}$$

and $X_1 - X_2 = Z_1 - Z_2$ with $Z_i = (\gamma_i^{1/2} - \gamma_f^{1/2})W_0 \in L^2(\Omega, H^1(M))$.

Lemma 6.15. Assume that $\gamma \in C(\mathbb{R}, \Sigma_f)$ is a solution to (2). Then there exists a probability space Ω and a Gaussian variable $X \in Y + C(\mathbb{R}, L^2(\Omega, H^1(M)))$ solution of (1) and of covariance γ .

Proof. Let $\varphi = \rho_\gamma$ and let X be the solution to

$$i\partial_t X = (-\Delta + \varphi)X$$

with initial datum X_0 a Gaussian variable with covariance γ_0 . Its covariance γ_X is the unique solution in $\gamma_f + \mathcal{L}(H^1)$ to the linear equation

$$i\partial_t \gamma_X = [-\Delta + \varphi, \gamma_X] \quad (19)$$

with initial datum $\gamma_{X_0} = \gamma(t=0)$.

Indeed, if an operator $\tilde{\gamma}$ belongs to Σ_f then it also belongs to $\gamma_f + \mathcal{L}(H^1)$.

What is more, if γ_1 and γ_2 are two solutions of (19) in $\gamma_f + \mathcal{L}(H^1)$ with the same initial datum then $\gamma_1 - \gamma_2$ is a solution to (19) in $\mathcal{L}(H^1)$ with initial datum 0.

Let $U(t)$ the flow of the linear equation in H^1 on $u : i\partial_t u = (-\Delta + \varphi)u$. The map U is an invertible. Hence, since,

$$i\partial_t(U(t)^*(\gamma_1 - \gamma_2)U(t)) = 0$$

the equation $i\partial_t \gamma_X = [-\Delta + \varphi, \gamma_X]$ has a unique solution.

Since γ is also a solution to $i\partial_t \gamma_X = [-\Delta + \varphi, \gamma_X]$ with initial datum $\gamma(t=0)$ we get that $\gamma_X = \gamma$, thus $\mathbb{E}(|X|^2) = \rho_\gamma$, which ensures that X is a solution to (1) and thanks to 6.13, a Gaussian solution to (1). \square

6.2.3 Global well-posedness

Corollary 6.16 (of Proposition 3.2). Let Ψ_f be the map from Σ_f to $C(\mathbb{R}, \Sigma_f)$ such that $\Psi_f(t)(\gamma_0) = \gamma_{Y_f(t)+Z(t)}$ where $Z(t)$ is the solution to (10) with initial datum Z_0 the Gaussian random process with covariance operator $(\gamma_0^{1/2} - \gamma_f^{1/2})^2$. The map Ψ_f is well-defined, it defines a solution to (2) and it is continuous for the distance d_f . Besides, $\Psi_f(t)\gamma_0$ is the unique solution to (2) with initial datum γ_0 .

Proof. We take $Z_0 = (\gamma_0^{1/2} - \gamma_f^{1/2})W_0$. It belongs to $L^2(\Omega, H^1(M))$ and $X_0 = Y_f(t=0) + Z_0$. Thanks to Proposition 3.2 we get a solution X of (1), and thanks to Proposition 1.1, we get a solution γ to the equation (2). Hence $\Psi_f(t)$ is well-defined. It is continuous in time and in the initial datum for the following reason : the distance between $\gamma_{X_1(t)}$ and $\gamma_{X_2(t)}$ is controlled by the norm of $X_1(t) - X_2(t)$, which is controlled by the norm of $Z_1(t) - Z_2(t)$. The continuity of the solution $Z(t)$ in both time and initial datum gives the result.

For the uniqueness of the solution, let γ_1 and γ_2 be two solutions of (2) with the initial datum γ_0 . For $i = 1, 2$, there exists $X_i(t)$ a solution of (1) which is a Gaussian variable of covariance γ_i . For $i = 1, 2$, $X_i(t=0)$ is a Gaussian variable of covariance γ_0 . Hence $X_1(t=0)$ and $X_2(t=0)$ have the same law. Therefore $X_1(t)$ and $X_2(t)$ too, which ensures that $\gamma_1 = \gamma_2$ and hence the uniqueness of the solution of (2). \square

Remark 6.6. *One could rewrite the corollary 6.16 as : the equation (18) is globally well-posed in Σ_f .*

6.3 On the focusing case

We consider the equation

$$i\partial_t \gamma = [-\Delta - \rho_\gamma, \gamma] \quad (20)$$

on \mathbb{R}^d , $d = 2, 3$.

Corollary 6.17 (of Proposition 5.4). *If γ_0 is such that $\text{Tr}((1 - \Delta)\gamma_0) < \infty$, $\int_{\mathbb{R}^d} |x|^2 \rho_{\gamma_0}(x) dx < \infty$ and*

$$\frac{1}{2} \text{Tr}((1 - \Delta)\gamma_0) - \frac{1}{4} \int_{\mathbb{R}^d} \rho_{\gamma_0}(x)^2 < 0$$

then there is at least one solution of (20) that exists locally in time and blows up at finite time in the sense that there exists T such that

$$\text{Tr}((1 - \Delta)\gamma(t)) \rightarrow \infty$$

when $t \rightarrow T$.

Proof. This is due to the fact that with X_0 the Gaussian random field of covariance γ_0 and $X(t)$ the solution of (16), we have

$$\text{Tr}((1 - \Delta)\gamma(t)) = \|X(t)\|_{L^2(\Omega, H^1(\mathbb{R}^d))}^2$$

and

$$\int_{\mathbb{R}^d} |x|^2 \rho_{\gamma_0}(x) dx = V(t = 0)$$

and

$$\frac{1}{2} \text{Tr}((1 - \Delta)\gamma_0) - \frac{1}{4} \int_{\mathbb{R}^d} \rho_{\gamma_0}(x)^2 = \mathcal{E}(X_0).$$

□

Remark 6.7. *One could rewrite the corollary 6.16 as : there exist blow-up solutions to the equation (20).*

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